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A tracking MPC formulation that is locally equivalent to economic MPC☆

Mario Zanon [a,*], Sébastien Gros [a], Moritz Diehl [b]

[a] Department of Signals and Systems, Chalmers University of Technology, Horsalsvagen 11, SE-41296 Goteborg, Sweden
[b] Department of Microsystems Engineering IMTEK, University of Freiburg, Georges-Koehler-Allee 102, 79110 Freiburg, Germany

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ABSTRACT

The stability proof for economic Model Predictive Control (MPC) is in general difficult to establish. In contrast, tracking MPC has well-established and practically applicable stability guarantees, but can yield poor closed-loop performance in terms of the selected economic criterion. In this paper, we derive a formal procedure to design a tracking MPC scheme so as to locally approximate the behaviour of economic MPC. Given an economic stage cost, the desired tracking stage cost can therefore be computed automatically. Because tracking MPC guarantees stability of the closed-loop system, our procedure succeeds if and only if economic MPC is locally stabilising. This fact can be used to certify whether economic MPC is not stabilising. We illustrate the theoretical developments in a simulated example.

1. Introduction

Classical MPC schemes are based on quadratic cost functions, and aim at minimizing the deviation of the system states and inputs from a given reference. This reference is often selected as a steady-state optimal operating point with respect to a known economic performance index. In contrast, Economic Model Predictive Control (EMPC) is based on directly optimizing the given economic performance index. As a result, economic MPC schemes usually outperform tracking MPC schemes especially when the system operates in transients.

The stability theory of tracking MPC is well developed and understood, see e.g. [1,2]. However, the stability theory of economic MPC is a relatively new field of research and many questions are still open. It initially considered linear systems and convex objectives [3,1]. For nonlinear systems, an analysis of average performance bounds was proposed in [4] and average constraints were considered in [5]. Lyapunov stability of economic MPC was first proven in [6] under a strong duality assumption and generalized in [7,8] under a strict dissipativity assumption. The necessity of strict dissipativity for optimal steady-state operation has been analyzed in [9,10]. A stability proof in the absence of terminal constraints is provided in [11]. The extension of the stability results to periodic systems has been considered in [12–14]. Economic MPC schemes where stability is enforced without the need of strict dissipativity have been proposed in [15–19]. Note that, by using the latter approaches, enforcing stability typically entails a loss of economic optimality.

In the nonlinear case, the strict dissipativity condition can be hard to verify, thus making it difficult to ensure the closed-loop stability of the economic MPC scheme. This paper proposes a strategy to compute a positive-definite tracking stage cost for nonlinear MPC (NMPC) so as to yield an NMPC feedback law that is locally equivalent to the one delivered by the economic MPC scheme. In [20], economic linear MPC has been analysed in the case of no active constraints at steady state and a method has been proposed for computing a positive definite stage cost for tracking MPC having locally the same behaviour as economic MPC. In this paper, we generalise these results to the case of active constraints at steady state and nonlinear tracking MPC formulations. Moreover, we prove that the obtained tracking MPC schemes locally approximate the behaviour of economic MPC up to first order.

This paper is structured as follows. In Section 2 we introduce the notation and describe the considered problem. In Section 3, we present the main result of the paper and a sketch of how it will be proven in the following sections. In Section 4 we prove the local equivalence of economic NMPC and an ad-hoc formulated indefinite linear MPC. In Section 5 the main result of [20] is briefly summarised, i.e. every stabilising linear MPC scheme with an indefinite stage cost and without active constraints at steady state can be reformulated as a positive definite linear MPC scheme. Moreover,

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* Corresponding author.
E-mail address: zmario@chalmers.se (M. Zanon).

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an alternative formulation for convexifying the stage cost is presented. In Section 6, the case of active constraints at steady state is analysed, and a convexification procedure is proposed. In Section 7 the results are extended to cover tracking NMPC. An example which illustrates the theoretical developments of this contribution is provided in Section 8. Conclusions and outlines for future research are provided in Section 9.

2. Problem description

This paper is concerned with time-invariant nonlinear discrete-time systems $x_{k+1} = f(x_k, u_k)$ that shall be operated such that constraints $h(x_k, u_k) \geq 0$ are satisfied and the cost $\sum_{k=0}^{\infty} \ell(x_k, u_k)$ is minimised. For notational simplicity and without loss of generality we assume that $\ell(0, 0) = 0$ and that $x_s = 0$, $u_s = 0$ is an optimal steady state, i.e. it solves the steady state problem

$$\min_{x,u} \quad \ell(x, u) \qquad \text{s.t.} \quad x - f(x, u) = 0, \qquad h(x, u) \geq 0. \tag{1}$$

We define $w_s = (x_s, u_s) = (0, 0)$, denote by λ_s and μ_s the optimal Lagrange multipliers associated with the equality and inequality constraints of problem (1) respectively. Note that these multipliers are in general nonzero.

Because the infinite horizon problem is computationally intractable, MPC approximates the infinite horizon problem by optimizing over a finite horizon N. We lump all states and controls in a vector $w = (w_0, w_1, \ldots, w_{N-1}, w_N)$, with $w_k = (x_k, u_k)$, $k = 0, \ldots, N-1$ and $w_N = x_N$. In the following, we will refer to the optimal solution by adding to the variable as a superscript the equation number of the problem it refers to, i.e. $w^{(2)}$ is the optimal primal solution of Problem (2). In the following, we introduce two approaches towards the aim of constrained economic optimisation.

The first approach is *tracking MPC*, where, at each time step, given the current state $\hat{x}_0$, one solves the following optimal control problem

$$\min_{w} \quad \sum_{k=0}^{N-1} \ell^{t}(x_k, u_k) + V_{f}^{t}(x_N) \tag{2a}$$

$$\text{s.t.} \quad x_0 - \hat{x}_0 = 0, \tag{2b}$$

$$x_{k+1} - f(x_k, u_k) = 0, \qquad k = 0, \ldots, N-1, \tag{2c}$$

$$h(x_k, u_k) \geq 0, \qquad k = 0, \ldots, N-1, \tag{2d}$$

$$x_N \in \mathbb{X}_{f}. \tag{2e}$$

The stage and terminal cost satisfy $\ell^{t}(x, u) \geq \alpha(\|x\|)$ for all feasible u and $V_{f}^{t}(x) \geq \alpha(\|x\|)$, where α is a $\mathcal{K}_\infty$ function [21]. Typically, the tracking stage and terminal cost are chosen as quadratic functions. In the remainder of the paper, whenever we refer to tracking MPC, we will implicitly assume such a choice. We define the optimal primal solution as $w^{(2)}(\hat{x}_0) = (x_0^{(2)}, u_0^{(2)}, \ldots, x_N^{(2)})$ and the tracking MPC feedback as $u^{t}(\hat{x}_0) = u_0^{(2)}$.

An alternative approach to tracking MPC is *economic MPC*, where at each time step, given the current state $\hat{x}_0$, one solves the following optimal control problem

$$\min_{w} \quad \sum_{k=0}^{N-1} \ell(x_k, u_k) + V_{f}(x_N) \tag{3a}$$

$$\text{s.t.} \quad (2b) - (2e) \tag{3b}$$

where we define the optimal primal solution as $w^{(3)}(\hat{x}_0) = (x_0^{(3)}, u_0^{(3)}, \ldots, x_N^{(3)})$ and the economic MPC feedback as $u^{e}(\hat{x}_0) = u_0^{(3)}$.

We remark that the terminal cost and constraint in both MPC Problems ought to be chosen together with the prediction horizon in order to guarantee stability [21,2]. For a given stage cost, this choice can be made so as to provide a good approximation of the infinite-horizon problem. For $\mathbb{X}_{f} = \{x_s\}$ we also define the Lagrangian of the economic MPC problem as

$$\mathcal{L}(w, \lambda, \mu) = \sum_{k=0}^{N-1} \ell(x_k, u_k) - \lambda_{k+1}^{\top}(x_{k+1} - f(x_k, u_k)) - \mu_k^{\top} h(x_k, u_k) - \lambda_0^{\top}(x_0 - \hat{x}_0) - \lambda_{N+1}^{\top}(x_N - x_s),$$

and we denote the optimal Lagrange multipliers as $\lambda_k^{(3)}$ and $\mu_k^{(3)}$. Throughout this paper we assume that the minimiser of Problems (1)–(3) exists.

The main difference between economic and tracking MPC is that the former typically outperforms the latter during transients. However, proving stability for economic MPC is much more involved than for tracking MPC. The difficulty stems from the fact that, in general, $\ell(x, u) \not\geq \alpha(\|x\|)$. In this paper, whenever we label a problem as economic we assume that $\nexists \alpha \in \mathcal{K}_\infty$ s.t. $\ell(x, u) \geq \alpha(\|x\|)$. Stability proofs for economic MPC typically rely on the concept of *rotated cost*. Given a function $\lambda(x)$, we define the rotated stage cost as

$$L(x_k, u_k) := \ell(x_k, u_k) + \lambda(x_k) - \lambda(f(x_k, u_k)). \tag{4}$$

Many developments in this paper can be interpreted using this concept of rotation or a generalisation that we will propose. We will refer to a problem as rotated whenever its stage cost is rotated and we will call *linear rotation* the one which uses a linear storage function. For more details on rotated cost and stability proofs for economic MPC, we refer to [6,8,10,14] and Appendix A.

Another difference between economic and tracking MPC regards the applicable algorithms: efficient numerical schemes for fast real-time NMPC, based on the generalised Gauss–Newton method, can in general only be applied to tracking MPC. For these reasons, in this paper, we aim at computing a positive definite stage cost for tracking (N)MPC such that it is locally first-order equivalent to economic (N)MPC. We state next the main result of this paper.

3. Main result

We introduce first the following key definitions of equivalent MPC problems.

Definition 1 *(Equivalent problems)*. Consider two MPC problems A and B and any initial state $\hat{x}_0$ for which both problems are feasible and have a unique solution satisfying linear independence constraint qualification (LICQ) and second order sufficient conditions (SOSC). Denote the optimal feedback laws as $u^{A}(\hat{x}_0)$ and $u^{B}(\hat{x}_0)$ respectively. We define the following:

(i) MPC problems A and B are *equivalent* iff they deliver the same feedback law, i.e. for all $\hat{x}_0$ it holds that $u^{A}(\hat{x}_0) = u^{B}(\hat{x}_0)$;
(ii) MPC problems A and B are *locally equivalent* iff there exists a neighbourhood $\mathcal{N}$ of the origin such that $u^{A}(\hat{x}_0) = u^{B}(\hat{x}_0)$;
(iii) MPC problems A and B are *locally first-order equivalent* iff there exists a neighbourhood $\mathcal{N}$ of the origin such that for all $\hat{x}_0 \in \mathcal{N}$ it holds that $\|u^{A}(\hat{x}_0) - u^{B}(\hat{x}_0)\| = O(\|\hat{x}_0\|^2)$.

Equivalence (i) implies local equivalence (ii) which, in turn, implies first-order local equivalence (iii). We will apply equivalence (i) and (ii) to convex problems and only equivalence (iii) will be applied to nonconvex problems. Throughout this paper, we will assume that all considered MPC problems are locally stabilising and satisfy SOSC, such that $u^{A,B}(0) = 0$ is unique and $u^{A,B}(\hat{x}_0)$ is unique for $\hat{x}_0$ in a neighbourhood of the origin, but could still be multi-valued for $\hat{x}_0$ far from the origin. For this reason, considering only the case of unique solutions is not very restrictive and allows us to establish local equivalence of nonconvex problems.

The main result of this paper is then expressed in the following theorem.

Theorem 2. *Given any stabilising economic NMPC of the form (3) satisfying the mild technical assumptions of Theorem 16 in Appendix B, it is possible to formulate both a tracking linear MPC and a tracking nonlinear MPC with a positive-definite stage cost which are locally first-order equivalent to the economic NMPC problem (3), i.e.* $\|u^{\rm t}(\hat{x}_0)-u^{\rm e}(\hat{x}_0)\| = O(\|\hat{x}_0\|^2)$*, for all* $\hat{x}_0$ *in a neighbourhood of the origin.*

Because of the complexity of the analysis, we will first establish some intermediate results. The proof of Theorem 2 will then be given in Section 7. In order to establish it, we will (a) formulate an economic linear MPC (ELMPC) scheme which is locally first-order equivalent to economic NMPC (ENMPC), (b) convexify its stage cost to obtain a locally equivalent linear MPC problem with positive definite cost (PD LMPC) and (c) formulate a locally first-order equivalent tracking NMPC problem (PD NMPC). This is summarised by the following scheme:

$$\text{ENMPC}\quad(3)\underset{4}{\overset{\text{(iii)}}{\leftrightarrow}}\text{ELMPC}\quad(6)\underset{5-6}{\overset{\text{(i)/(ii)}}{\leftrightarrow}}\text{PDLMPC}\quad(22)\underset{7}{\overset{\text{(iii)}}{\leftrightarrow}}\text{PDNMPC}\quad(2),$$

where we denote equivalence in the sense of Definition 1 by using the symbol $\leftrightarrow$ with the type of equivalence as superscript. The reference to the section in which the equivalence is proven is delivered as a subscript. In order to clearly distinguish between linear and nonlinear MPC, we denote them as LMPC and NMPC respectively. Moreover, we define as ELMPC a linear MPC problem with quadratic indefinite stage cost. By definition, the solution of the PD LMPC problem, if it exists, is unique. Therefore, by equivalence (i)/(ii) also the solution of problem ELMPC must be unique. Problems ENMPC and PD NMPC, instead, can have multiple solutions as they are nonconvex.

Throughout this paper, we will assume that the reference economic MPC scheme (3) is locally stabilising. Note that, whenever a positive-definite stage cost yielding local first-order equivalence of tracking and economic MPC does not exist, we do certify that economic MPC can not stabilising. In this case, operating the system at steady state rather than e.g. periodically, is not optimal. We leave the analysis of such situation for future research and recall that steady state operation can be enforced by e.g. [20, Remark 4]. As we will prove in Sections 5 and 6, the formal design of the locally first-order equivalent tracking MPC scheme involves solving a convex SDP. Note that these computations are done offline and the online computational burden is the one of a standard tracking MPC scheme.

In the following, we will construct an ELMPC scheme which is first-order locally equivalent to the ENMPC scheme.

4. Locally first-order equivalent ENMPC and ELMPC

In this section, we analyse the properties that enforce ENMPC $(3)\overset{\text{(iii)}}{\leftrightarrow}$ELMPC (6). First, we establish an important property relating the economic MPC Problem (3) to the steady-state Problem (1).

Lemma 3. *Consider a stabilising economic MPC formulation of the form (3) with* $V(x)$ *the cost-to-go of Problem (3) with an infinite horizon. Assume that (a) the horizon is infinite, or (b) the gradient of the terminal cost* $V_{\rm f}(\cdot)$ *satisfies* $\nabla V_{\rm f}|_{x=x_{\rm s}} = \lambda_{\rm s}$ *and* $\nabla^2 V_{\rm f} \geq \nabla^2 V$*, or (c) the terminal constraint is* $x_N = x_{\rm s}$*. Then, the Lagrange multipliers* λ_k, μ_k *of the MPC problem solved for the initial state* $\hat{x}_0 = x_{\rm s}$ *coincide with those of the steady state problem* $\lambda_{\rm s}$, $\mu_{\rm s}$.

Proof. The proof is directly obtained by comparing the KKT conditions of the MPC problem and the steady-state problem. Because the economic MPC problem is stabilising, the primal solution is $x_k = x_{\rm s}$ and $u_k = u_{\rm s}$. Then, by replacing $\lambda_k = \lambda_{\rm s}$ and $\mu_k = \mu_{\rm s}$, one obtains that the KKT conditions are satisfied. □

Note that the result holds both for the original and the rotated stage cost, provided that the economic MPC problem and the steady-state problem are formulated using the same stage cost.

In order to formulate the locally first-order equivalent ELMPC problem, we define

$$H := \nabla^2_{w_k}\mathcal{L}_k(w,\lambda,\mu), \qquad q := \nabla_{w_k}\ell(x_k,u_k), \tag{5a}$$

$$A := \nabla_{x_k} f(x_k,u_k)^\top, \qquad B := \nabla_{u_k} f(x_k,u_k)^\top, \tag{5b}$$

$$C := \nabla_{w_k} h(x_k,u_k)^\top, \tag{5c}$$

where all expressions are evaluated at the primal-dual solution of the steady-state Problem (1), i.e. $w_{\rm s}$, $\lambda_{\rm s}$, $\mu_{\rm s}$. Moreover, we used the fact that the Lagrangian Hessian of Problem (3) is block diagonal, with each block given by

$$\nabla^2_{w_k}\mathcal{L}_k(w,\lambda,\mu) := \nabla^2\ell(x_k,u_k) + \sum_{i=0}^{n_x}\lambda_{k+1,i}\nabla^2 f_i(x_k,u_k) - \sum_{j=0}^{n_\mu}\mu_{k,j}\nabla^2 h_j(x_k,u_k).$$

Theorem 4. *The ELMPC problem*

$$\min_{\substack{x_0,\ldots,x_N\\ u_0,\ldots,u_{N-1}}} \quad \sum_{k=0}^{N-1}\frac{1}{2}\begin{bmatrix}x_k\\u_k\end{bmatrix}^\top H\begin{bmatrix}x_k\\u_k\end{bmatrix} + q^\top\begin{bmatrix}x_k\\u_k\end{bmatrix} \tag{6a}$$

$$\text{s.t.}\quad x_0 - \hat{x}_0 = 0, \tag{6b}$$

$$x_{k+1} - Ax_k - Bu_k = 0, \quad k=0,\ldots,N-1, \tag{6c}$$

$$C\begin{bmatrix}x_k\\u_k\end{bmatrix} + e \geq 0, \quad k=0,\ldots,N-1, \tag{6d}$$

$$x_N = 0, \tag{6e}$$

is locally first-order equivalent to the ENMPC problem (3).

Proof. The equivalence is a direct consequence of Lemma 17 in Appendix B. □

Note that, because one can rotate the cost without changing the primal solution, a new formulation is easily obtained which is also first-order locally equivalent to ENMPC (3). Because it is simpler to formulate and analyse, a scheme with no gradient in the cost (i.e. with $q = 0$) can be of interest. We will establish next that such a formulation exists only if there are no active constraints at steady state and can be obtained by rotating the cost using $\lambda(x) = \lambda_{\rm s}^\top x$.

Lemma 5. *Suppose that the steady-state Problem (1) has no active constraints. Then the rotated cost* $L(x, u)$ *with* $\lambda(x) = \lambda_{\rm s}^\top x$ *has zero gradient at steady state. If there are strictly active constraints at steady state, then the following holds:*

$$\frac{\partial L(x_k,u_k)}{\partial w_k}|_{w=0} = \mu_{\rm s}^\top\frac{\partial h(x_k,u_k)}{\partial w_k}|_{w=0}.$$

Proof. In the case of no active constraints at steady-state, the rotated cost coincides with the Lagrangian of the optimisation Problem (1) evaluated at the optimal Lagrange multipliers, which, by definition of optimality, has zero gradient at the optimum. If instead some constraints are active at steady state, the desired equality is directly obtained by writing the KKT conditions of Problem (1). □

We just proved that one can rotate the cost by exploiting the equality constraints. Therefore, rotating the cost using active inequality constraints might sound appealing. However, we prove next that in that case the obtained problem would not deliver the same primal solution as the original one.

Lemma 6. *Consider vectors $\bar{\lambda}$ and $\bar{\mu}$, a vector of fixed parameters θ, and the two parametric NLPs*

$$\min_{w} \quad \bar{f}(w,\theta) \quad \text{s.t.} \quad \bar{g}(w,\theta)=0, \quad \bar{h}(w,\theta)\geq 0. \tag{7}$$

$$\min_{w} \quad \bar{f}(w,\theta)-\bar{\lambda}^\top\bar{g}(w,\theta)-\bar{\mu}^\top\bar{h}(w,\theta) \quad \text{s.t.} \quad \bar{g}(w,\theta)=0,$$
$$\bar{h}(w,\theta)\geq 0. \tag{8}$$

We will call NLP (7) the original NLP and NLP (8) the rotated NLP. For NLP (7), let us introduce the optimal Lagrange multipliers $\lambda^{(7)}(\theta)$, $\mu^{(7)}(\theta)$ associated with the equality and inequality constraints, respectively. Equivalently, for NLP (8), we introduce $\lambda^{(8)}(\theta)$ and $\mu^{(8)}(\theta)$. Then, the original and rotated NLPs (7) and (8) deliver the same primal solution for all θ iff $\bar{\mu}=0$. Moreover, it holds that $\lambda^{(8)}(\theta)=\lambda^{(7)}(\theta)-\bar{\lambda}$ and $\mu^{(8)}(\theta)=\mu^{(7)}(\theta)$.

Proof. The proof is obtained by writing the KKT conditions for the two NLPs. The detailed developments are given in Appendix C. □

An intuitive explanation of Lemma 6 can be obtained as follows. Consider rotating the cost with respect to a strictly active inequality constraint, so as to remove the corresponding gradient from the cost. This makes the constraint weakly active so that Problems (7) and (8) do not deliver the same primal solution for all parameter values. Indeed, because the new multiplier becomes zero, the cost of Problem (8) becomes locally insensitive to relaxing or tightening the constraint. On the contrary, this does not hold for Problem (7), as the multiplier is nonzero.

In the following two sections, we will state the conditions for ELMPC (6) $\overset{\text{(i)/(ii)}}{\leftrightarrow}$ PDLMPC (22). We will analyse first the case of no active constraints at steady state and we will prove that the equivalence can be obtained by means of a quadratic rotation. Afterwards, we will consider the case of active constraints at steady-state, which requires a more involved analysis.

5. Linear MPC without active constraints at steady state

The equivalence ELMPC (6) $\overset{\text{(i)/(ii)}}{\leftrightarrow}$ PDLMPC (22) has been analysed in [20] for the case of no active constraints at steady state. In the following, we first summarise the main result of [20], which states this equivalence. Then, we extend that result by proposing a new method for computing the positive definite stage cost. This also makes it possible to prove the necessity of strict dissipativity with a quadratic storage function for stability of LMPC in the case of no active constraints at steady state.

5.1. Review of previous results

Let us consider the following LMPC problem where, without loss of generality, we assume that the linearly rotated stage cost with zero gradient at steady-state is used

$$\min_{w} \quad \sum_{k=0}^{N-1}\frac{1}{2}\begin{bmatrix}x_k\\u_k\end{bmatrix}^\top \underbrace{\begin{bmatrix}Q & S^\top\\ S & R\end{bmatrix}}_{=H\not\succ 0}\begin{bmatrix}x_k\\u_k\end{bmatrix}+\frac{1}{2}x_N^\top P_N x_N \tag{9a}$$

$$\text{s.t.} \quad x_0-\hat{x}_0=0, \tag{9b}$$

$$x_{k+1}-Ax_k-Bu_k=0, \quad k=0,\ldots,N-1, \tag{9c}$$

$$C\begin{bmatrix}x_k\\u_k\end{bmatrix}+e\geq 0, \quad k=0,\ldots,N-1. \tag{9d}$$

Here, we assume that $e>0$, such that no constraint is active at steady state and the gradient of the stage cost at the origin is 0. The prediction horizon N and terminal cost matrix P_N are tuning parameters to be adequately selected, and are not discussed in this paper. For more details on the topic we refer to e.g. [1,2].

Note that, for the case of $H\succ 0$, the equivalence is directly obtained, and we label as Economic LMPC (ELMPC) the case of $H\not\succ 0$. While in general it is preferable to formulate LMPC problems using a positive definite stage cost both for ensuring closed-loop stability and for computational reasons, the use of indefinite stage costs does not necessarily result in an unstable closed-loop behaviour [22].

Theorem 7. *Suppose that both (a) the economic LQR formulated using matrices A, B, H exists and (b) the economic LMPC scheme (9) is stabilizing. Then, there exist (a) an LQR with positive definite cost which yields the same feedback matrix as the economic LQR and (b) a positive definite LMPC scheme which yields the same primal solution as the economic LMPC scheme (9).*

Proof. The proof is given in [20]. □

Theorem 7 implies that for all stabilising schemes the stage and terminal cost matrices H, $P_N\not\succ 0$ can always be replaced by appropriately selected matrices $\tilde{H}$, $\tilde{P}_N\succ 0$ without changing the (primal) solution of the LMPC scheme. In this paper, we will denote by the term *convexification* the procedure of computing the positive definite stage cost.

We define the feedback matrix K and the cost-to go matrix P as the ones obtained from the LQR formulated using the original cost. Note that the LQR does not necessarily exist, in which case the infinite horizon economic problem is not stabilising, and the system is not optimally operated at steady state. The case of non optimal steady state operation is out of the scope of this paper and is the object of ongoing research. Because the choice of $\tilde{P}$, $\tilde{H}$ is not unique, it was proposed in [20] to compute them by solving the following SDP

$$\min_{\tilde{P},\tilde{Q},\tilde{R},\tilde{S},\tilde{H}} \quad \|\tilde{H}-I\|^2 \tag{10a}$$

$$\text{s.t.} \quad \tilde{H}=\begin{bmatrix}\tilde{Q} & \tilde{S}^\top\\ \tilde{S} & \tilde{R}\end{bmatrix} \tag{10b}$$

$$\tilde{H}\succeq 0 \tag{10c}$$

$$\tilde{P}\succeq 0 \tag{10d}$$

$$\tilde{Q}+A^\top\tilde{P}A-\tilde{P}-(\tilde{S}^\top+A^\top\tilde{P}B)K=0, \tag{10e}$$

$$(\tilde{R}+B^\top\tilde{P}B)K-(\tilde{S}+B^\top\tilde{P}A)=0, \tag{10f}$$

$$\tilde{R}+B^T\tilde{P}B=R+B^TPB, \tag{10g}$$

$$\tilde{S}+B^T\tilde{P}A=S+B^TPA. \tag{10h}$$

The convexification of the terminal cost matrix is given by $\tilde{P}_N=P_N-P+\tilde{P}$.

In the following, we propose a new formulation where fewer optimisation variables are necessary and the feedback matrix K does not need to be computed explicitly.

5.2. Further analysis of the steady-state case with no active constraints

Let us define $\delta P = P - \tilde{P}$ and introduce the linear operator

$$\mathcal{H}(\delta P) = \begin{bmatrix} A^\top \delta P A - \delta P & A^\top \delta P B \\ B^\top \delta P A & B^\top \delta P B \end{bmatrix}. \tag{11}$$

We then define $\tilde{H} = H + \mathcal{H}(\delta P)$. It can be easily checked that this construction directly enforces Constraints (10e)–(10h). Then, one can convexify the cost by solving the following SDP which is equivalent to (10)

$$\min_{\delta P} \quad \|H + \mathcal{H}(\delta P) - I\|^2 \tag{12a}$$

$$\text{s.t.} \quad H + \mathcal{H}(\delta P) \succeq 0. \tag{12b}$$

Note that $\tilde{H} \succeq 0$ implies that the cost-to-go matrix $\tilde{P}$ of the LQR computed using $\tilde{H}$ satisfies $\tilde{P} \succeq 0$. Moreover, no knowledge about the cost-to-go P or the feedback gain matrix K is needed in order to compute the stage cost convexification given by $\tilde{H} \succeq 0$. Note that the cost function of the SDP is somewhat arbitrary and has been chosen to favour well-conditioned stage cost matrices. However, other choices are possible and even preferable. They will be discussed in Section 6.2.

This new formulation of the cost convexification also highlights the connection with the concept of strict dissipativity and rotated cost. In the following, we prove an important implication of Theorem 7 on the necessity of strict dissipativity. Note that necessity of strict dissipativity has been proven in [10]. However, Corollary 8 states that the storage function can be selected to be quadratic for linear systems with quadratic stage cost.

Corollary 8. *Consider an economic LMPC formulation of the form (9) with no active constraints at steady state. Strict dissipativity with a quadratic storage function is not only sufficient but also necessary for the stability of the closed-loop system.*

Proof. By Theorem 7, if the economic LMPC (9) with no active constraints at steady state is stabilising, then there exists a matrix δP such that $\tilde{H} := H + \mathcal{H}(\delta P) \succ 0$. It can be seen that the cost convexification (11) is a nonlinear cost rotation of the form (4) with storage function $\lambda(x_k) = -x_k^\top \delta P x_k$. Therefore, the convexified stage cost satisfies

$$L(x, u) := \ell(x, u) + \lambda(x) - \lambda(Ax + Bu) \; = \begin{bmatrix} x_k \\ u_k \end{bmatrix}^\top \tilde{H} \begin{bmatrix} x_k \\ u_k \end{bmatrix} \geq \epsilon \|x\|^2,$$

with $\epsilon > 0$. As a result, strict dissipativity is therefore not only sufficient, but also necessary for closed-loop stability of economic LMPC with no active constraints at steady state.□

Note that the original stability proof based on strict dissipativity [8] considers states and controls to be in a compact set $\mathbb{Z}$. In the framework of LMPC with no active constraints at steady state, however, we obtain the stronger result of Corollary 8. Moreover, Eq. (11) establishes that the convexification procedure is a nonlinear rotation of the cost. We remark that in the economic LMPC Formulation (9) we assumed without loss of generality that the cost was linearly rotated. Then, in the general case, for economic LMPC without active constraints at steady state, the rotation of the cost is given by the composition of a linear rotation and a convexification, i.e. $\lambda(x) = \lambda_s^\top x - x^\top \delta P x$. In the following section, we turn to the case of active constraints at steady state.

6. Linear MPC with active constraints at steady state

In this section, we study the equivalence ELMPC (6) $\overset{\text{(i)/(ii)}}{\leftrightarrow}$ PDLMPC (22) in the case some constraints are active at steady state. First, we establish a theoretical framework for proving the existence of the equivalent PD LMPC formulation. Subsequently, we provide a practical approach for computing its stage cost.

6.1. Equivalent LMPC formulations with active constraints at steady state

In the case of a linear system, a quadratic cost and strictly active constraints at steady state, the cost can in general have a non-zero gradient at the optimum even after a linear rotation of the stage cost (cf. Lemma 6). We consider an LMPC problem formulated with the stage cost rotated linearly by using the Lagrange multipliers corresponding to the system dynamics evaluated at the origin, given by

$$\min_{w} \quad \sum_{k=0}^{N-1} \frac{1}{2} \begin{bmatrix} x_k \\ u_k \end{bmatrix}^\top H \begin{bmatrix} x_k \\ u_k \end{bmatrix} + q^\top \begin{bmatrix} x_k \\ u_k \end{bmatrix} \tag{13a}$$

$$\text{s.t.} \quad x_0 - \hat{x}_0 = 0, \tag{13b}$$

$$x_{k+1} - Ax_k - Bu_k = 0, \quad k = 0, \ldots, N-1, \tag{13c}$$

$$C \begin{bmatrix} x_k \\ u_k \end{bmatrix} + e \geq 0, \quad k = 0, \ldots, N-1, \tag{13d}$$

$$x_N = 0. \tag{13e}$$

We consider a terminal point constraint for simplicity, however, any formulation satisfying the conditions of Lemma 3 can be used in our derivation.

We analyse now some local properties of Problem (13), which hold for a given fixed active set. Note that the problem formulations we will consider can be too restrictive and inadequate for practical applications. However, they are helpful for establishing some properties that can then be exploited in practice. We denote the set of indices relative to the strictly active constraints at steady state as $\mathbb{A}_s$ and we assume the Jacobian of the strictly active constraints $C_{\mathbb{A}_s}$ to be full row rank. Note that, the steady state being $x_s = 0$, $u_s = 0$, the affine term in the active inequality constraints must be zero, i.e. $e_{\mathbb{A}_s} = 0$.

Theorem 9. *Let us consider the region $\mathbb{X}_0$ of initial states $\hat{x}_0$ around the origin for which the optimal active set of Problem (13) includes all constraints which are strictly active for $\hat{x}_0 = 0$. Assume that the LMPC problem*

$$\min_{w} \quad \sum_{k=0}^{N-1} \frac{1}{2} \begin{bmatrix} x_k \\ u_k \end{bmatrix}^\top H \begin{bmatrix} x_k \\ u_k \end{bmatrix} + q^\top \begin{bmatrix} x_k \\ u_k \end{bmatrix} \tag{14a}$$

$$\text{s.t.} \quad x_0 - \hat{x}_0 = 0, \tag{14b}$$

$$x_{k+1} - Ax_k - Bu_k = 0, \quad k = 0, \ldots, N-1, \tag{14c}$$

$$C_{\mathbb{A}_s} \begin{bmatrix} x_k \\ u_k \end{bmatrix} = 0, \quad k = 0, \ldots, N-1, \tag{14d}$$

$$x_N = 0. \tag{14e}$$

is stabilising for all $\hat{x}_0 \in \mathbb{X}_0$. Then there exist matrices δP and F such that

$$H + \mathcal{H}(\delta P) + C_{\mathbb{A}_s}^\top F C_{\mathbb{A}_s} \succ 0.$$

Additionally,

$Z^\top(H+\mathcal{H}(\delta P))Z \succ 0,$

holds for any Z with $C_{\mathbb{A}_s}Z=0$, $Z^\top Z=I$ and $\begin{bmatrix} C_{\mathbb{A}_s}^\top & Z \end{bmatrix}$ invertible.

Proof. For any matrix F, Problem (14) can be reformulated as

$$\min_{w} \quad \sum_{k=0}^{N-1} \frac{1}{2}\begin{bmatrix} x_k \\ u_k \end{bmatrix}^\top \underbrace{(H+C_{\mathbb{A}_s}^\top F C_{\mathbb{A}_s})}_{\bar{H}} \begin{bmatrix} x_k \\ u_k \end{bmatrix} + q^\top \begin{bmatrix} x_k \\ u_k \end{bmatrix} \tag{15a}$$

$$\text{s.t.} \quad (14\text{b})-(14\text{e}). \tag{15b}$$

By construction, for any $\hat{x}_0 \in \mathbb{X}_0$, Problem (15) delivers the same solution as Problem (14). Moreover, because matrix $C_{\mathbb{A}_s}$ is full row rank, there exists a matrix $F \succ 0$, such that $C_{\mathbb{A}_s}\bar{H}C_{\mathbb{A}_s}^\top \succ 0$.

We consider now a series of problems which are in general not equivalent, but which share some important properties for $\hat{x}_0=0$, and which will allow us to prove the theorem. We therefore now focus on the special case $\hat{x}_0=0$. In this setting, Lyapunov stability of Problem (14) entails that the primal solution of Problem (15) is zero, i.e. $w^{(15)}(0)=w^{(14)}(0)=0$. In the following, we consider a problem formulation in which we dualise the constraints, i.e. we replace the original cost with the Lagrangian of the problem. By Lemma 3, the Lagrange multipliers of Problem (15) with $\hat{x}_0=0$ coincide with those of the original steady-state Problem (1). Moreover, the primal solution is not changed by the following linear rotation:

$$J = \sum_{k=0}^{N-1} \frac{1}{2}\begin{bmatrix} x_k \\ u_k \end{bmatrix}^\top \bar{H} \begin{bmatrix} x_k \\ u_k \end{bmatrix} + q^\top \begin{bmatrix} x_k \\ u_k \end{bmatrix}$$

$$-\lambda_s^\top x_0 + \sum_{k=0}^{N-1} -\lambda_s^\top (x_{k+1}-Ax_k-Bu_k) - \mu_s^\top C_{\mathbb{A}_s}\begin{bmatrix} x_k \\ u_k \end{bmatrix} \tag{16}$$

$$= \sum_{k=0}^{N-1} \frac{1}{2}\begin{bmatrix} x_k \\ u_k \end{bmatrix}^\top \bar{H} \begin{bmatrix} x_k \\ u_k \end{bmatrix}, \tag{17}$$

where, by optimality, all linear terms cancel out. Note that our new cost is given by the Lagrangian function of Problem (15) evaluated at the optimal Lagrange multipliers.

By using (17) as a cost in the LMPC, we obtain the following problem

$$\min_{w} \quad \sum_{k=0}^{N-1} \frac{1}{2}\begin{bmatrix} x_k \\ u_k \end{bmatrix}^\top \bar{H} \begin{bmatrix} x_k \\ u_k \end{bmatrix} \tag{18a}$$

$$\text{s.t.} \quad (14b)-(14e). \tag{18b}$$

For the initial condition $\hat{x}_0=0$, the unique primal solution is $x_k^{(18)}=0$ and $u_k^{(18)}=0$. We recall that, by an appropriate choice of matrix F, we obtain $C_{\mathbb{A}_s}\bar{H}C_{\mathbb{A}_s}^\top \succ 0$ and, because the primal solution is zero, the Lagrange multipliers associated with the path constraints are also zero, i.e. $\mu^{(18)}(0)=0$. As a result, one can remove them without changing the primal solution for $x_0=\hat{x}_0$. Then, also the following problem

$$\min_{w} \quad \sum_{k=0}^{N-1} \frac{1}{2}\begin{bmatrix} x_k \\ u_k \end{bmatrix}^\top \bar{H} \begin{bmatrix} x_k \\ u_k \end{bmatrix} \tag{19a}$$

$$\text{s.t.} \quad x_0-\hat{x}_0=0, \tag{19b}$$

$$x_{k+1}-Ax_k-Bu_k=0, \quad k=0,\ldots,N-1, \tag{19c}$$

$$x_N=0, \tag{19d}$$

delivers the unique primal solution $x_k^{(19)}=0$ and $u_k^{(19)}=0$ for $\hat{x}_0=0$. Because this property holds for any arbitrarily long horizon length N, it must also hold for an infinite horizon. As we prove in Lemma 18 given in Appendix D, this entails that the infinite horizon problem must deliver a stabilising feedback matrix. Therefore, by Theorem 7, the desired property is obtained, i.e.:

$$\exists \ \delta P,\ F \ \text{ s.t.} \quad H+\mathcal{H}(\delta P)+C_{\mathbb{A}_s}^\top F C_{\mathbb{A}_s} \succ 0. \tag{20}$$

By definition of the nullspace Z, it also holds that

$$\exists \ \delta P, \ \text{ s.t.} \quad Z^\top\left(H+\mathcal{H}(\delta P)\right)Z \succ 0. \tag{21}$$

□

Remark 10. The terminal point constraint should be handled with care, as it might lead to infeasibility if the horizon is shorter than the amount of states of the system, i.e. $N<n_x$. Moreover, extra care needs to be taken if the system is not controllable but stabilisable. The developments of Theorem 9 directly extended to the case of stabilisable systems and a stabilising terminal cost.

By relying on the result of Theorem 9, we can now formulate the LMPC problem with positive definite stage cost as

$$\min_{w} \quad \sum_{k=0}^{N-1} \frac{1}{2}\begin{bmatrix} x_k \\ u_k \end{bmatrix}^\top \tilde{H} \begin{bmatrix} x_k \\ u_k \end{bmatrix} + q^\top \begin{bmatrix} x_k \\ u_k \end{bmatrix} \tag{22a}$$

$$\text{s.t.} \quad x_0-\hat{x}_0=0, \tag{22b}$$

$$x_{k+1}-Ax_k-Bu_k=0, \quad k=0,\ldots,N-1, \tag{22c}$$

$$C\begin{bmatrix} x_k \\ u_k \end{bmatrix} + e \geq 0, \quad k=0,\ldots,N-1, \tag{22d}$$

$$x_N=0. \tag{22e}$$

with $\tilde{H}=H+\mathcal{H}(\delta P)+C_{\mathbb{A}_s}^\top F C_{\mathbb{A}_s}$. We are now ready to formulate the main result of this section in the following theorem.

Theorem 11. *Consider any linear MPC scheme (13) which is stabilising in the sense of Lyapunov with region of attraction $\mathcal{X}\subseteq\mathbb{X}_0$ and which satisfies the mild technical assumptions of Theorem 16 in Appendix B. Then, the positive definite LMPC scheme (22) delivers the same primal solution as the original problem for initial conditions $\hat{x}_0$ in a neighbourhood of the origin, i.e. $w^{(22)}(\hat{x}_0)=w^{13}(\hat{x}_0)$ holds$\forall \ \hat{x}_0\in\bar{\mathbb{X}}_0$, where we define the set $\bar{\mathbb{X}}_0$ as the set of initial conditions $\hat{x}_0$ for which the set of active constraints for both Problem (13) and (22) coincides and includes all constraints that are active for $\hat{x}_0=0$.*

Moreover, if the positive definite stage cost matrix can be computed using $F=0$, then the equivalence holds for all initial conditions $\hat{x}_0$, i.e. $w^{(22)}(\hat{x}_0)=w^{(13)}(\hat{x}_0), \quad \forall \ \hat{x}_0$.

Proof. Given the stage cost matrix H, Theorem 9 guarantees the existence of a positive definite stage cost matrix $\tilde{H}=H+\mathcal{H}(\delta P)+C_{\mathbb{A}_s}^\top F C_{\mathbb{A}_s}$. We now distinguish two cases.

Let us first consider the case $F=0$. Because the convexification (11) is a rotation in the sense of [8], the addition of the term $\mathcal{H}(\delta P)$ to the stage cost does not change the primal solution of the problem, regardless of the active set. Then, for $F=0$, the equivalence of Problems (13) and (22) is directly obtained, regardless of the active set.

For $F\neq 0$, the equivalence of Problems (13) and (22) will in general depend on the active set. If the set $\mathbb{A}_k$ of the indices of the active constraints at each stage k is such that $\mathbb{A}_s\subseteq\mathbb{A}_k$, then the optimal cost is independent of the choice of F. Indeed, by denoting the nullspace of the active constraints Jacobian as Z_k, such that $C_{\mathbb{A}_k}Z_k=0$, we obtain $Z_k^\top C_{\mathbb{A}_s}^\top F C_{\mathbb{A}_s} Z_k=0$. Therefore, if for each stage k it holds that $\mathbb{A}_s\subseteq\mathbb{A}_k$, Problem (22) delivers the same primal solution as the original Problem (13). Moreover, Theorem 16 guarantees that there exists a neighbourhood of initial conditions $\hat{x}_0$ around the origin in which the constraints that are strictly active for $\hat{x}_0 = 0$ remain active. Finally, $w^{(22)}(\hat{x}_0) = w^{(13)}(\hat{x}_0)$ for all $\hat{x}_0 \in \bar{\mathbb{X}}_0$, implies that the control laws of the two MPC schemes are identical, i.e. $u^{(22)}(\hat{x}_0) = u^{(13)}(\hat{x}_0)$.□

6.2. A practical approach for convexifying the stage cost

The convexification procedure can be formulated as the following Linear Matrix Inequality (LMI)

$$\tilde{H} = H + \mathcal{H}(\delta P) + C_{\mathbb{A}_s}^\top F C_{\mathbb{A}_s} \succ 0. \tag{23}$$

Note that, similarly to the unconstrained case, by solving the LMI (23) in variables δP and F the stage cost matrix $\tilde{H}$ can be badly conditioned. Moreover, it is interesting to seek solutions for which $F = 0$, such that the equivalence ELMPC $(6) \overset{(i)/(ii)}{\leftrightarrow}$ PDLMPC (22) holds for all feasible initial conditions.

In order to mitigate the problem of bad conditioning, one can minimise the condition number of matrix $\tilde{H}$. While this problem seems nonconvex, there exist convex formulations. In this paper, however, we are mostly interested in positive definiteness of $\tilde{H}$. A simple convex approach for approximately minimising the condition number consists in minimising $\gamma\beta - \alpha$, with $\gamma \geq 0$ a parameter of choice and $\tilde{H} \succeq \alpha I$, $\beta I \succeq \tilde{H}$. Note that the condition number is given by $\kappa = \beta/\alpha$ and its first order Taylor expansion around $(\alpha, \beta) = (a, b)$ yields $\kappa \approx \frac{b}{a} + \frac{1}{a}\beta - \frac{b}{a^2}\alpha$. One can therefore interpret the proposed cost function as the cost of a subproblem of a sequential convex programming (SCP) method for minimising the condition number.

In order to also address the second observation, we propose a two-step procedure in which it is first attempted to solve the convexification problem using $F = 0$ and, only when necessary, $F \neq 0$ is used. We construct the convexification procedure using the following SDP:

$$\min_{\delta P, F, \alpha, \beta} \quad \gamma\beta - \alpha + \rho\|F\| \tag{24a}$$

$$\text{s.t.} \quad H + \mathcal{H}(\delta P) + \eta C_{\mathbb{A}_s}^\top F C_{\mathbb{A}_s} \succeq \alpha I, \tag{24b}$$

$$H + \mathcal{H}(\delta P) + \eta C_{\mathbb{A}_s}^\top F C_{\mathbb{A}_s} \preceq \beta I, \tag{24c}$$

with $\gamma \geq 0$, $\rho \geq 0$ two tuning parameters and $\|\cdot\|$ an appropriately chosen matrix norm. Parameter $\eta \in \{0, 1\}$ is used to construct the aforementioned 2-step procedure. First, SDP (24) is solved using $\eta = 0$, $\rho > 0$, which results in $F = 0$. If the solution satisfies $\alpha^* > 0$, we have found the stage cost matrix for a positive definite reformulation of the original problem which is globally equivalent to it. If even with $\gamma = 0$ we obtain $\alpha^* < 0$, we set $\eta = 1$ and solve the SDP again. In this second case the equivalence of the positive definite and original problems will only be local.

For the second step of the convexification procedure, one can decide to set $\rho = 0$ and constrain δP to match the one found by the previous solution. This way, the second solution only acts on matrix F in order to make the stage cost matrix positive definite and reduce its condition number.

Note that, if one chooses $\gamma = 0$, it might be necessary to introduce the additional constraint $\alpha \leq \bar{\alpha}$ with $\bar{\alpha}$ a parameter of choice. This ensures that the SDP is bounded.

7. Formal design of the tracking (N)MPC based on economic criteria

In this section, we state the conditions for the equivalence PDLMPC $(22) \overset{(iii)}{\leftrightarrow}$ PDNMPC (2) to hold. By relying on the equivalence result of Lemma 17 it is possible to formulate a tracking NMPC problem whose QP approximation at the origin is given by Problem (22), so that the tracking NMPC problem is locally first-order equivalent to the economic NMPC problem, i.e. in the sense of (iii). However, some difficulties can be encountered if this last equivalence needs to be satisfied by using a positive definite stage cost also in the tracking NMPC formulation. We explain in the following why this is not straightforward and we propose a way to tackle this problem.

We recall that the stage-cost matrix $\tilde{H}$ of PD LMPC (22) is given by (20):

$$\tilde{H} = \nabla^2_{w_k}\mathcal{L}_k(w, \lambda, \mu) + \mathcal{H}(\delta P) + C_{\mathbb{A}_s}^\top F C_{\mathbb{A}_s} \succ 0.$$

Then,for the PD NMPC, the choice of tthe quadratic cost

$$\sum_{k=0}^{N-1} w_k^\top \hat{H}_t w_k + q_t^\top w_k,$$
$$\hat{H}_t = \nabla^2\ell(x_s, u_s) + \mathcal{H}(\delta P) + C_{\mathbb{A}_s}^\top F C_{\mathbb{A}_s}, \qquad q_t = \nabla\ell(x_s, u_s),$$

yields the desired local first-order equivalence. However, such a choice does not guarantee positive definiteness of the stage cost, as

$$\hat{H}_t = \underbrace{\tilde{H}}_{\succ 0} + (\underbrace{\sum_{i=0}^{n_x}\lambda_s\nabla^2 f_i(x_k, u_k)}_{\not\succeq 0} - \underbrace{\sum_{j=0}^{n_\mu}\mu_s\nabla^2 h_j(x_k, u_k)}_{\not\succeq 0}).$$

In order to tackle the problem relative to the Hessian of the equality constraints, one ought to consider the rotated version of the economic NMPC problem, so that $\lambda_s = 0$, therefore $\sum_{i=0}^{n_x}\lambda_s\nabla^2 f_i(x_k, u_k) = 0$. However, as proven in Lemma 6, it is not possible to rotate the cost with respect to the inequality multipliers without changing the primal solution, so that one needs to address the problem of the contribution of the path constraints to the Hessian.

In the case $\tilde{H} \not\succ -\sum_{j=0}^{n_\mu}\mu_{k,j}\nabla^2_w h_j(x_k, u_k)$, we propose to tackle the problem by introducing slack variables s and replacing the inequality constraints $\bar{h}(w) \geq 0$ by the equality constraints $\bar{h}(w) - s = 0$ and inequality constraints $s \geq 0$. Then, the inequality constraints do not contribute to the Lagrangian Hessian because they are linear. At the same time, the contribution to the Hessian which was previously due to inequality constraints is now due to equality constraints which can be rotated. This reasoning is formalised in the following Lemma for generic parametric NLP formulations.

Lemma 12. *Consider vectors $\bar{\lambda}$ and $\bar{\mu}$, a vector of fixed parameters θ, and the two parametric NLPs*

$$\min_{w,s} \quad \bar{f}(w, \theta) \tag{25a}$$

$$\text{s.t.} \quad \bar{g}(w, \theta) = 0, \qquad \bar{h}(w, \theta) - s = 0, \qquad s \geq 0. \tag{25b}$$

and

$$\min_{w,s} \quad \bar{f}(w, \theta) - \bar{\lambda}^\top \bar{g}(w, \theta) - \bar{\mu}^\top(\bar{h}(w, \theta) - s) \tag{26a}$$

$$\text{s.t.} \quad \bar{g}(w, \theta) = 0, \qquad \bar{h}(w, \theta) - s = 0, \qquad s \geq 0. \tag{26b}$$

For NLP (25), let us introduce the Lagrange multipliers associated with $\bar{g}(w, \theta) = 0$ as $\lambda^{(25)}(\theta)$, those associated with $\bar{h}(w, \theta) - s = 0$ as $\mu^{(25)}(\theta)$ and those associated with $s \geq 0$ as $\nu^{(25)}$. Equivalently, for NLP (26), we introduce $\lambda^{(26)}(\theta)$, $\mu^{(26)}(\theta)$ and $\nu^{(26)}(\theta)$. Then, both NLP (25) and (26) deliver the same primal solution for all $\bar{\lambda}$, $\bar{\mu}$. Moreover, it holds that $\lambda^{(26)}(\theta) = \lambda^{(25)}(\theta) - \bar{\lambda}$, $\mu^{(26)}(\theta) = \mu^{(25)}(\theta) - \bar{\mu}$ and $\nu^{(26)}(\theta) = \nu^{(25)}(\theta)$.

Proof. As we only rotate equality constraints, Lemma 6 directly applies. □

The proposed slack reformulation of the economic NMPC Problem (3) has the useful property that, after rotation of the cost, none of the constraints contributes to the Hessian of the Lagrangian for an initial state close to the origin. Note that the proposed rotation is different from the one proposed in both [6,8]. We are now ready to prove the main result of this paper, i.e. Theorem 2.

Proof *(Theorem 2)*. One can always formulate Problem (3) using the proposed slack variable Formulation (26). Then, none of the inequality constraints contributes to the Lagrangian Hessian at steady state. The existence of the positive definite tracking formulation is then a direct consequence of Lemma 12 and Theorem 11. Indeed, one can rotate the economic NMPC NLP in the form (26) using $\tilde{\lambda} = \lambda_s$ and $\tilde{\mu} = \mu_s$. The corresponding locally equivalent linear MPC problem is obtained as Problem (6), which has a possibly indefinite stage cost. Note that the slack variables introduced by the formulation (26) can be treated as dummy controls which do not affect the system dynamics. Theorem 11 then states that there also exists a locally equivalent positive definite linear MPC Formulation (22). Moreover, as we use Formulation (26), the inequality constraints do not contribute to the Lagrangian Hessian. Therefore, by using the same stage cost as in (22), one can directly formulate a tracking NMPC scheme (2) which has the same local behaviour as the economic NMPC problem.

Using Theorem 16 and Lemma 17 given in Appendix B we obtain that

$$\underbrace{\frac{\partial w^{(6)}}{\partial \hat{x}_0}|_{\hat{x}_0=0}}_{\text{Indefinite MPC}} = \underbrace{\frac{\partial w^{(3)}}{\partial \hat{x}_0}|_{\hat{x}_0=0}}_{\text{EMPC}} \quad \text{and} \quad \underbrace{\frac{\partial w^{(2)}}{\partial \hat{x}_0}|_{\hat{x}_0=0}}_{\text{PD tracking NMPC}} = \underbrace{\frac{\partial w^{(22)}}{\partial \hat{x}_0}|_{\hat{x}_0=0}}_{\text{PD MPC}}.$$

Moreover, Theorem 11 proves that $w^{(22)} = w^{(6)}$ locally holds. Therefore

$$\underbrace{\frac{\partial w^{(22)}}{\partial \hat{x}_0}|_{\hat{x}_0=0}}_{\text{PDMPC}} = \underbrace{\frac{\partial w^{(6)}}{\partial \hat{x}_0}|_{\hat{x}_0=0}}_{\text{IndefiniteMPC}},$$

and we obtain the desired equivalence:

$$\|u^{t}(\hat{x}_0) - u^{e}(\hat{x}_0)\| = O(\|\hat{x}_0\|^2).$$

□

Remark 13. It is important to remark that, in order to formulate the tracking NMPC problem which locally approximates economic NMPC, we first rotate the stage cost of economic NMPC. This implies that, as stated in Lemma 5, the gradient of the rotated cost in the origin is given by:

$$\frac{\partial L(x_k, u_k)}{\partial w_k}|_{w=0}^{\top} = \frac{\partial h(x_k, u_k)}{\partial w_k}|_{w=0}^{\top} \mu_s.$$

Note that, because we formulate the economic NMPC Problem (3) in its slack form, i.e. as Problem (26), we have $h(x_k, u_k) = s_k$ where s_k are the slack variables added as ficticious controls in vector u_k.

8. Numerical example

In this section, we present a numerical example which illustrates the theoretical concepts presented in the previous sections. We remark that the scope of this section is not to assess how well a tuned tracking NMPC scheme can perform, but rather to show that the tuning procedure allows one to formulate an NMPC scheme whose closed loop behaviour is closer than that of a generic NMPC to the one obtained by the economic NMPC scheme, which also results in an improved performance. Clearly, the economic performance of the tuned tracking NMPC scheme can be extremely good for some applications, while for some other applications it can still be rather poor, as all derived results concern local properties. As opposed to the economic NMPC formulation, however, the tuned tracking NMPC formulation does always provide stability guarantees. Moreover, more efficient and faster numerical schemes are available for tracking formulations.

Table 1
Model parameters. The units are omitted and are consistent with the physical quantities they correspond to.

a	b	c	d	e	f	g	h
0.5616	0.3126	48.43	0.507	55	0.1538	55	0.16
M	C	UA_2	C_p	λ	λ_s	F_1	X_1
20	4	6.84	0.07	38.5	36.6	10	5 %
F_3	T_1	T_{200}					
50	40	25					

In order to compare the performance of the different MPC schemes, we use the following performance measure

$$G = \frac{P_{\text{eco}} - P_{\text{track}}}{\sum_{k=0}^{T-1} P_s}, \tag{27}$$

where P_{eco} and P_{track} are the cumulative costs of economic and tracking NMPC respectively over T time steps and P_s is the steady-state optimal cost.

Let us consider an evaporation process that removes a volatile species from a solvent. The model equations are given by [23]

$$M\dot{X}_2 = F_1X_1 - F_2X_2, \qquad C\dot{P}_2 = F_4 - F_5, \tag{28}$$

where

$$\begin{aligned}
&T_2 = aP_2 + bX_2 + c, && T_3 = dP_2 + e, && \lambda F_4 = Q_{100} - F_1C_p(T_2 - T_1)\\
&T_{100} = fP_{100} + g, && Q_{100} = UA_1(T_{100} - T_2), && UA_1 = h(F_1 + F_3),\\
&F_{100} = \frac{Q_{100}}{\lambda_s}, && Q_{200} = \frac{UA_2(T_3 - T_{200})}{1 + UA_2/(2C_pF_{200})}, && \lambda F_5 = Q_{200},\\
&F_2 && = F_1 - F_4,
\end{aligned}$$

and the states are $x = (X_2, P_2)$ the controls are $u = (P_{100}, F_{200})$. The model parameters are given in Table 1. The economic objective is given by

$$\ell(x, u) = 10.09(F_2 + F_3) + 600F_{100} + 0.6F_{200}.$$

The system is subject to the following constraints

$$\begin{aligned}
&X_2 \geq 25\,\%, && 40\,\text{kPa} \leq P_2 \leq 80\,\text{kPa},\\
&P_{100} \leq 400\,\text{kPa}, && F_{200} \leq 400\,\text{kg/min}.
\end{aligned}$$

The optimal steady state is given by

$$x_s = (25,\ 49.743), \qquad u_s = (191.713,\ 215.888). \tag{29}$$

We consider a sampling time $t_s = 1$ s and formulate the NMPC scheme using a piecewise constant control parametrisation, a prediction horizon $N = 200$ and the terminal point constraint $x_N = x_s$. In order to avoid feasibility problems we reformulate the concentration constraint using a slack variable s (introduced as a ficticious control) so that the constraint reads $X_2 + s \geq 25\,\%$, $s \geq 0$, and the cost is given by

$$\ell(x, u) = 10.09(F_2 + F_3) + 600F_{100} + 0.6F_{200} + 10^3 s.$$

A purely economic NMPC scheme has been compared in a simulation scenario to a conventional tracking NMPC implementation, the proposed economically tuned NMPC scheme and a simplified version of the tuned NMPC scheme which uses a diagonal weighting matrix. On a 300 s long simulation, a pressure disturbance $\Delta P_2 = 1$ kPa is applied to the system at time instants $t_0 = 0$ s, $t_1 = 20$ s, and $t_2 = 40$ s. For the tracking schemes we choose the following “normal” stage cost matrix (denoted by subscript n), we compute

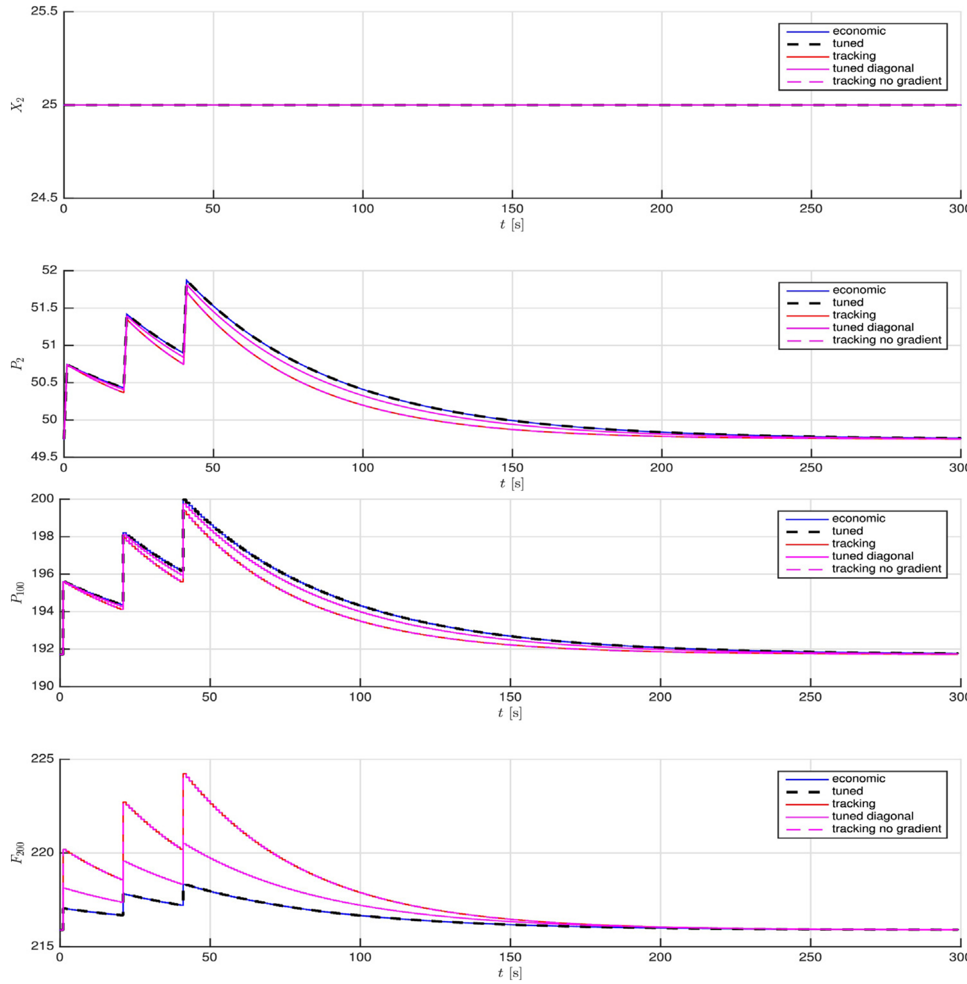

Fig. 1. States and controls for the evaporation process. Blue = EMPC, dashed black = tuned tracking, red = normal tracking, magenta = tuned diagonal, dashed magenta = normal tracking with zero gradient. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of the article.)

the following tuned stage cost matrix (denoted by subscript t) and its diagonal version (denoted by subscript td)

$$H_{\rm n} = \mathrm{diag}([\,10 \quad 10 \quad 0.1 \quad 0.1 \quad 0.1\,]),$$
$$H_{\rm t} = \begin{bmatrix} 6.96\cdot 10^{0} & -7.42\cdot 10^{-1} & 1.54\cdot 10^{-1} & -9.55\cdot 10^{-4} & 0 \\ -7.42\cdot 10^{-1} & 1.23\cdot 10^{-1} & -1.62\cdot 10^{-2} & 6.86\cdot 10^{-5} & 0 \\ 1.54\cdot 10^{-1} & -1.62\cdot 10^{-2} & 7.93\cdot 10^{-3} & -2.10\cdot 10^{-5} & 0 \\ -9.55\cdot 10^{-4} & 6.86\cdot 10^{-5} & -2.10\cdot 10^{-5} & 4.53\cdot 10^{-3} & 0 \\ 0 & 0 & 0 & 0 & 1.49\cdot 10^{1} \end{bmatrix}, \quad H_{\rm td} = \mathrm{Diag}(H_{\rm t}),$$

where the units of the weights are chosen consistently with the physical quantities they correspond to, so as to yield a dimensionless cost. We remark that, in order to be able to compute a convex tuned tracking stage cost, we need to make use of the term $C_{\mathbb{A}_s}^\top F C_{\mathbb{A}_s}$. The operator diag constructs a diagonal matrix with the diagonal elements taken from the given vector and the operator Diag returns a matrix of the same size as the original with the diagonal elements are unchanged and all off-diagonal elements are set to 0.

The tracking cost has then been defined as

$$\ell_\bullet^{\rm t}(x,u) = \begin{bmatrix} x \\ u \end{bmatrix}^\top H_\bullet \begin{bmatrix} x \\ u \end{bmatrix} + \nabla_w \ell(x,u)|_{w_s}^\top \begin{bmatrix} x \\ u \end{bmatrix}, \qquad \bullet \in \{\mathrm{n}, \mathrm{t}, \mathrm{td}\},$$

where $w = \begin{bmatrix} x & u \end{bmatrix}^\top$. We remark that, as there are active constraints at steady state, the gradient of the cost at steady state can not be made equal to zero and needs to be included in the tracking cost formulation. For comparison, we also implemented a tracking NMPC formulation which uses the stage cost matrix $H_{\rm n}$ and zero gradient, i.e.

$$\ell_{\rm n,ng}^{\rm t}(x,u) = \begin{bmatrix} x \\ u \end{bmatrix}^\top H_{\rm n} \begin{bmatrix} x \\ u \end{bmatrix}.$$

The state and control trajectories resulting from the closed loop simulations are displayed in Fig. 1. It can be seen that the tuned tracking NMPC scheme is so close to the economic NMPC one that the trajectories are indistinguishable by eye inspection. The NMPC scheme formulated using the diagonal of the tuned stage cost matrix performs better than the normal though there is no a priori guarantee that this should be the case. The performance index for the three schemes reflects this situation:

$$G_{\rm n} = -3.2\cdot 10^{-4}, \qquad G_{\rm n,ng} = -1.5\cdot 10^{-3}, \qquad G_{\rm td} = -1.1\cdot 10^{-4}, \qquad G_{\rm t} = -1.2\cdot 10^{-7}.$$

The tuned NMPC scheme performs 3 orders of magnitude better than the two other tracking NMPC schemes. It is therefore clear

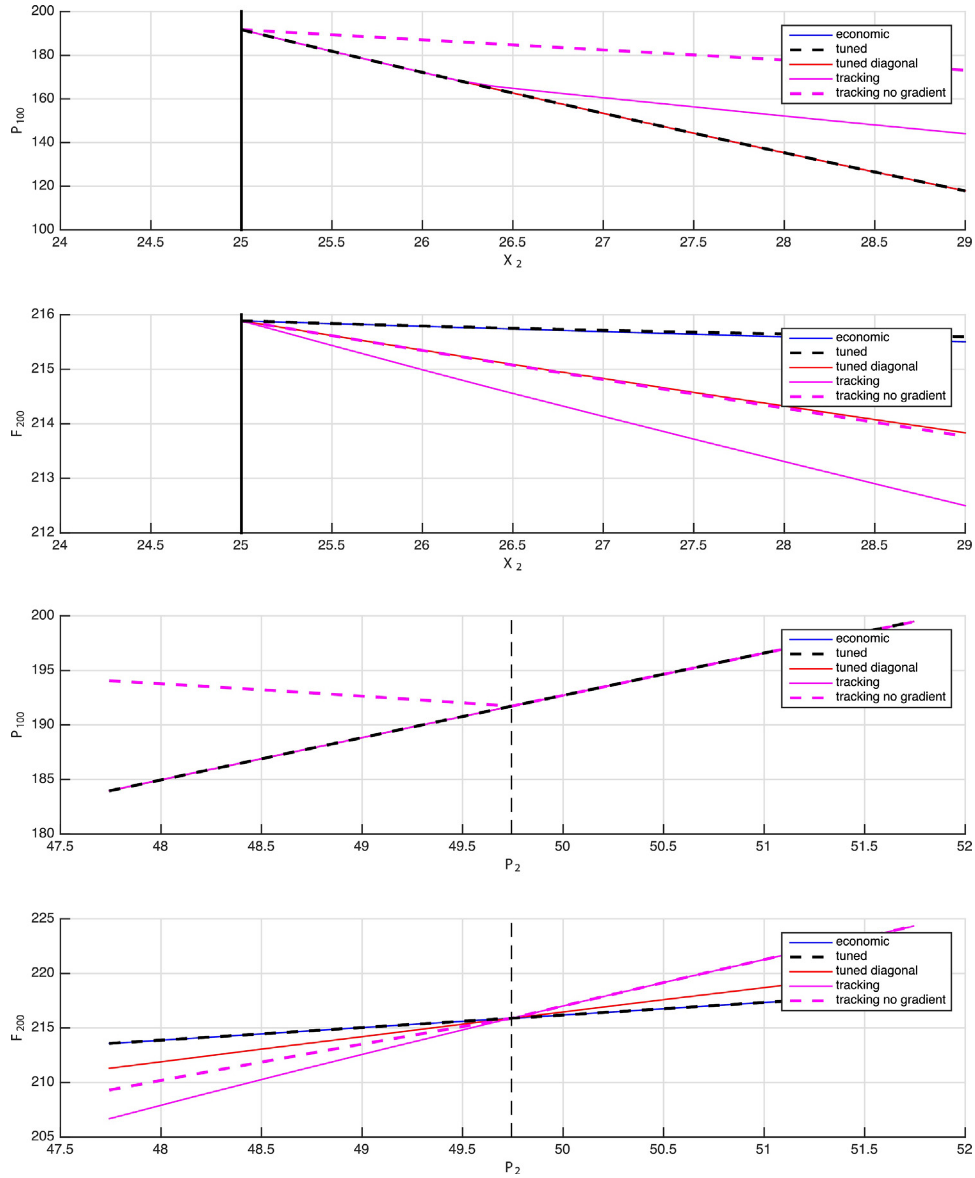

Fig. 2. MPC control law depending on initial conditions close to the optimal steady state. In the top graph we have perturbed X_2, while in the bottom one we have perturbed P_2. In all graphs, the optimal steady state is displayed in dashed black line and the bounds in thick continuous black line. Blue = EMPC, dashed black = tuned tracking, red = tuned diagonal, magenta = normal tracking, dashed magenta = normal tracking with zero gradient. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of the article.)

that the off-diagonal elements can be very important in order to have an accurate approximation of the economic NMPC behaviour.

In order to confirm the theoretical result that $\|u^{\mathrm{t}}(\hat{x}_0) - u^{\mathrm{e}}(\hat{x}_0)\| = O(\|\hat{x}_0\|^2)$, we have perturbed the initial condition in a neighborhood of the optimal steady state. The resulting control laws are displayed in Fig. 2, where it can be seen that the tuned tracking nonlinear MPC scheme is the only tracking scheme for which it holds that $\frac{\partial u^{\mathrm{t}}(\hat{x}_0)}{\partial \hat{x}_0}|_{\hat{x}_0=x_{\mathrm{s}}} = \frac{\partial u^{\mathrm{e}}(\hat{x}_0)}{\partial \hat{x}_0}|_{\hat{x}_0=x_{\mathrm{s}}}$.

From Fig. 2 we also deduce that the tracking NMPC scheme with zero gradient performs very differently from the others if the pressure P_2 drops. By running the same scenario with negative perturbations of the pressure, we obtain the following performance indices:

$$G_{\mathrm{n}} = -5.4 \cdot 10^{-6}, \quad G_{\mathrm{n,ng}} = -1.6 \cdot 10^{-1}, \quad G_{\mathrm{td}} = -1.1 \cdot 10^{-4}, \quad G_{\mathrm{t}} = -1.2 \cdot 10^{-7}.$$

All the tracking NMPC schemes with the correct gradient perform very well, while the tracking NMPC scheme with zero gradient performs much worse than all others.

Finally, on a 300 s long simulation, a concentration disturbance $\Delta X_2 = 1\%$ is applied to the system at time instants $t_0 = 0$ s, $t_1 = 20$ s, and $t_2 = 40$ s. We obtain the following performance indices:

$$G_{\mathrm{n}} = -2.6 \cdot 10^{-5}, \quad G_{\mathrm{n,ng}} = -4.2 \cdot 10^{-2}, \quad G_{\mathrm{td}} = -1.6 \cdot 10^{-5}, \quad G_{\mathrm{t}} = -7.4 \cdot 10^{-9}.$$

Again, all tracking NMPC schemes which make use of the non-zero gradient at the origin perform well compared to the economic NMPC, while the one which has zero gradient at the origin is by far the worst. We remark that this last tracking NMPC scheme is the only one which does not bring the concentration X_2 back to its steady state value in one step.

9. Conclusions

In this paper we have set a theoretical background for a formal design of the stage cost for both linear and nonlinear tracking MPC schemes approximating the behaviour of economic NMPC also in the presence of active constraints at the optimal steady state. We have proven that our design procedure yields tracking MPC schemes which locally deliver a first-order approximation of the economic MPC control law. A necessary condition for the tuning to exist is that the economic MPC scheme is locally stabilising.

When considering nonlinear tracking MPC in the presence of nonlinear inequality constraints active at steady state, the theoretical developments valid for linear tracking MPC do not apply. In order to tackle this issue, we have proposed a slack reformulation of the NMPC scheme which only has linear inequality constraints but is equivalent to the original formulation.

We have proposed a practical approach for computing the positive definite stage cost matrix for tracking NMPC. In order to compute a stage cost which is well conditioned, we formulate the problem as an SDP. Whenever our approach leads to an infeasible SDP, we have certified that the economic MPC scheme is not locally stabilising.

Finally, we have applied the theoretical developments to an example in simulations. We analyse both (a) the closed-loop behaviour in terms of trajectories and economic performance and (b) the feedback control law as a function of the initial state. All results demonstrate the beneficial effect of the proposed tuning procedure.

Acknowledgements

We want to thank Prof. Lars Grüne for inspiring discussions and helpful comments.

Appendix A.

We define an MPC scheme as *stabilising* if the closed-loop system is Lyapunov stable, which we define as follows.

Theorem 14 (Lyapunov stability). *Suppose that the set $\mathcal{X} \subset \mathbb{R}^{n_x}$ is positive invariant for the closed-loop system $x_{k+1} = f(x_k, u^\bullet(x_k))$, $\bullet \in \{$e, t$\}$, and that x_s lies in the interior of $\mathcal{X}$. If there exists a Lyapunov function $V : \mathcal{X} \to \mathbb{R}$ for the closed-loop system and the equilibrium x_s, then x_s is asymptotically stable with a region of attraction $\mathcal{X}$.*

For tracking MPC, Lyapunov stability is guaranteed by the condition $\ell^t(x, u) \geq \alpha(\|x\|)$ for all feasible u, in combination with continuity of the cost and system dynamics, an appropriate choice of the terminal cost, and the satisfaction of a technical controllability assumption. For economic MPC, the condition $\ell(x, u) \geq \alpha(\|x\|)$ is in general violated and the current stability theory relies on a strict dissipativity assumption to replace it.

Definition 15 *(Strict dissipativity [24]).* The system $x_{k+1} = f(x_k, u_k)$ is dissipative on a set $\mathbb{W} = \mathbb{W}_\mathbb{X} \times \mathbb{W}_\mathbb{U}$ with respect to the supply rate $\ell : \mathbb{W} \to \mathbb{R}$ if there exists a function $\lambda : \mathbb{W}_\mathbb{X} \to \mathbb{R}$, which is bounded from below on $\mathbb{W}_\mathbb{X}$ and such that the following inequality is satisfied for all $(x_k, u_k) \in \mathbb{W}$:

$$\lambda(f(x_k, u_k)) - \lambda(x_k) \leq \ell(x_k, u_k). \tag{A.1}$$

If there exists a positive definite function ρ such that for all $(x_k, u_k) \in \mathbb{W}$:

$$\lambda(f(x_k, u_k)) - \lambda(x_k) \leq -\rho(\|x_k\|) + \ell(x_k, u_k), \tag{A.2}$$

then the system is strictly dissipative on $\mathbb{W}$.

The assumption of strict dissipativity has been used in [8] in order to prove Lyapunov stability of economic MPC. The stability proof hinges on the rotated cost (4) satisfying $L(x_k, u_k) \geq \rho(\|x_k\|)$ with $\lambda(x)$ continuous in x_s and bounded. In [6] and [8], it has been proven that the MPC problem (3) formulated using the rotated cost delivers the same primal solution as the original one. Clearly, if strict dissipativity holds, $L(x_k, u_k) \geq \alpha(\|x_s\|)$ holds on a compact set so that the rotated economic MPC problem satisfies all the assumptions used to prove Lyapunov stability of tracking MPC. Because there exists no systematic method to find or dismiss the existence of a storage function $\lambda(x)$ such that the system satisfies strict dissipativity in the general case, it is hard to guarantee Lyapunov stability for economic MPC schemes.

Appendix B.

We provide next a result from parametric optimisation.

Theorem 16. *(Continuous differentiability [25,26]) Let us consider a parametric optimisation problem which depends on parameter t*

$$\min_w \quad \bar{f}(w,t) \qquad \text{s.t.} \quad \bar{g}(w,t) = 0, \qquad \bar{h}(w,t) \geq 0. \tag{B.1}$$

We define the Lagrangian as $\bar{\mathcal{L}}(w, t, \lambda, \mu) = \bar{f}(w, t) - \lambda^\top \bar{g}(w, t) - \mu^\top \bar{h}(w, t)$, the solution points depending on t as $(w^*(t), \lambda^*(t), \mu^*(t))$, and the set of indices of the active constraints for $t = 0$ as $\mathbb{A}$. Let us assume that the KKT point $(w^*(0), \lambda^*(0), \mu^*(0))$ satisfies linear independence constraint qualification (LICQ), strong second order sufficient conditions and strict complementarity. Let us moreover assume that the solution $(\delta w^*, \delta\lambda^*, \delta\mu^*_\mathbb{A})$ of the following quadratic program (QP)

$$\min_w \quad \frac{1}{2}\delta w^\top \nabla_w^2 \bar{\mathcal{L}} \delta w + \left(\frac{\partial}{\partial t}\nabla_w \bar{\mathcal{L}}\right)^\top \delta w \tag{B.2a}$$

$$\text{s.t.} \quad \nabla_t \bar{g} + \nabla_w \bar{g}^\top \delta w = 0, \tag{B.2b}$$

$$\nabla_t \bar{h}_\mathbb{A} + \nabla_w \bar{h}_\mathbb{A}^\top \delta w = 0, \tag{B.2c}$$

where all derivatives are evaluated at $(w^*(0), \lambda^*(0), \mu^*(0))$, satisfies the strict complementarity condition. Then

(i) *there exists an ϵ and a differentiable curve $v(t) = (w^*(t), \lambda^*(t), \mu^*(t))$ of KKT points that satisfy the optimality conditions for Problem (B.1), for $t \in (-\epsilon, \epsilon)$;*
(ii) *at $t = 0$ the one sided derivative of this curve is given by*

$$\frac{\partial v}{\partial t} = \lim_{t \to 0^+} \frac{1}{t}\begin{bmatrix} w^*(t) - w^*(0) \\ \lambda^*(t) - \lambda^*(0) \\ \mu^*(t) - \mu^*(0) \end{bmatrix} = \begin{bmatrix} \delta w^* \\ \delta\lambda^* \\ \begin{bmatrix} \delta\mu^*_\mathbb{A} \\ 0 \end{bmatrix} \end{bmatrix},$$

where we ordered μ^* such that the multipliers corresponding to the active constraints are followed by those corresponding to the inactive ones.

Proof. The proof is given in [26] and [25, Theorem 3.3.4 and Corollary 3.3.1]. □

This theorem is particularly important because (a) it ensures that there is an interval $(-\epsilon, \epsilon)$ inside which the set of strongly active constraints remains unchanged and (b) it states that $w^*(t) = w^*(0) + t\delta w^* + O(t^2)$, for $t \in (-\epsilon, \epsilon)$.

The original version of the theorem considered milder assumptions that allowed the existence of weakly active constraints for $t = 0$. In that case one can only prove the existence of the one-sided derivative of the solution curve, as $t = 0$ can be a point of non-differentiability of the solution curve. For simplicity, in this paper we only consider economic MPC formulations which do not have weakly active constraints at steady state. An extension to the case of weakly active constraints at steady state seems possible but requires a more detailed technical discussion which is left for future research.

An implication of Theorem 16 is given in the following lemma.

Lemma 17. *Consider any NLP of the form (B.1) which satisfies the assumption of Theorem 16 and the following QP*

$$\min_{w} \quad \frac{1}{\delta}2 w^\top \nabla_w^2 \bar{\mathcal{L}} \delta w + \left(\frac{\partial}{\partial t}\nabla_w \bar{\mathcal{L}}\right)^\top \delta w \tag{B.3a}$$

$$\text{s.t.} \quad \bar{g}(w^{(B.1)}, t) + \nabla_w \bar{g}^\top \delta w = 0, \tag{B.3b}$$

$$\bar{h}(w^{(B.1)}, t) + \nabla_w \bar{h}^\top \delta w \geq 0, \tag{B.3c}$$

where all derivatives are evaluated at the optimal point $w^*(t), t, \lambda^*(t), \mu^*(t)$. *Then the following holds at* $t = 0$:

$$\frac{\partial v^{(B.3)}}{\partial t} = \frac{\partial v^{(B.1)}}{\partial t},$$

i.e. the two problems locally deliver the same solution up to first-order.

Proof. The proof is obtained by noting that both Problem (B.1) and (B.3) deliver the same QP (B.2).□

Appendix C.

We provide next the proof of Lemma 6.

Proof *(Lemma 6).* Let us denote the optimal solution of Problem (7) by $w^*(\theta) = w^{(7)}(\theta)$. The KKT conditions of NLP (7) read

$$\nabla_w \bar{f}(w^*(\theta), \theta) - \nabla_w \bar{g}(w^*(\theta), \theta)\lambda^{(7)}(\theta) - \nabla_w \bar{h}(w^*(\theta), \theta)\mu^{(7)}(\theta) = 0, \tag{C.1a}$$

$$\bar{g}(w^*(\theta), \theta) = 0, \tag{C.1b}$$

$$\bar{h}(w^*(\theta), \theta) \geq 0, \tag{C.1c}$$

$$\mu^{(7)}(\theta) \geq 0, \tag{C.1d}$$

$$\mu_i^{(7)}(\theta)\bar{h}_i(w^*(\theta), \theta) = 0, \quad i = 1, \ldots, n_h. \tag{C.1e}$$

If the original and rotated NLPs deliver the same primal solution, then $w^*(\theta)$ must also be a KKT point for the rotated problem. The KKT conditions of NLP (8) then read

$$\nabla_w \bar{f}(w^*(\theta), \theta) - \nabla_w \bar{g}(w^*(\theta), \theta)(\lambda^{(8)}(\theta) + \bar{\lambda}) - \nabla_w \bar{h}(w^*(\theta), \theta)(\mu^{(8)}(\theta) + \bar{\mu}) = 0, \tag{C.2a}$$

$$\bar{g}(w^*(\theta), \theta) = 0, \tag{C.2b}$$

$$\bar{h}(w^*(\theta), \theta) \geq 0, \tag{C.2c}$$

$$\bar{\mu}^{(8)}(\theta) \geq 0, \tag{C.2d}$$

$$\bar{\mu}_i^{(8)}(\theta)\bar{h}_i(w^*(\theta), \theta) = 0, \quad i = 1, \ldots, n_h. \tag{C.2e}$$

Conditions (C.1a) and (C.2a) are equivalent for $\lambda^{(8)}(\theta) = \lambda^{(7)}(\theta) - \bar{\lambda}$ and for $\mu^{(8)}(\theta) = \mu^{(7)}(\theta) - \bar{\mu}$. However, we will now show that one can in general only satisfy Conditions (C.2d) and (C.2e) for all θ iff $\bar{\mu} = 0$. Let us consider the original problem and a variation $\delta\theta$ which yields $0 = \mu_i^{(7)}(\theta + \delta\theta) < \mu_i^{(7)}(\theta)$. In this case the only feasible choice which still satisfies (C.2d) is $\bar{\mu}_i \leq 0$. However, in case $\bar{h}_i(w^*(\theta + \delta\theta), \theta + \delta\theta) > 0$, then the only admissible choice is $\bar{\mu}_i = 0$, otherwise Condition (C.2e) cannot be satisfied.

Therefore, Problems (7) and (8) are always delivering the same primal solution if and only if $\bar{\mu} = 0$, i.e. $\mu^{(8)}(\theta) = \mu^{(7)}(\theta)$.□

Appendix D.

We prove that, under the conditions of Theorem 9, for system (A, B) the LQR formulated using the stage cost matrix $\bar{H}$ satisfying $C_{\mathbb{A}_s}\bar{H}C_{\mathbb{A}_s}^\top \succ 0$ delivers a stabilising feedback matrix.

Lemma 18. *Consider the infinite-horizon problem*

$$V^+(\hat{x}_0) := \min_{w} \quad \sum_{k=0}^{\infty} \frac{1}{2}\begin{bmatrix} x_k \\ u_k \end{bmatrix}^\top \bar{H} \begin{bmatrix} x_k \\ u_k \end{bmatrix} \tag{D.1a}$$

$$\text{s.t.} \quad x_0 - \hat{x}_0 = 0, \tag{D.1b}$$

$$x_{k+1} - Ax_k - Bu_k = 0, \quad k = 0, 1, \ldots, \tag{D.1c}$$

$$\lim_{N\to\infty} x_N = 0. \tag{D.1d}$$

If the system is stabilisable and if for $\hat{x}_0 = 0$ *the unique primal solution is* $x_k^{(D.1)} = 0$, $u_k^{(D.1)} = 0$, *then Problem (D.1) is stabilising for all initial conditions* $\hat{x}_0$.

Proof. Let us define the following helper problem

$$V^-(\hat{x}_0) := \min_{w} \quad \sum_{k=-\infty}^{-1} \frac{1}{2}\begin{bmatrix} x_k \\ u_k \end{bmatrix}^\top \bar{H} \begin{bmatrix} x_k \\ u_k \end{bmatrix} \tag{D.2a}$$

$$\text{s.t.} \quad x_0 - \hat{x}_0 = 0, \tag{D.2b}$$

$$x_{k+1} - Ax_k - Bu_k = 0, \quad k = 0, 1, \ldots, \tag{D.2c}$$

$$\lim_{N\to\infty} x_{-N} = 0, \tag{D.2d}$$

We then define

$$V(\hat{x}_0) := \min_{w} \quad \sum_{k=-\infty}^{\infty} \frac{1}{2}\begin{bmatrix} x_k \\ u_k \end{bmatrix}^\top \bar{H} \begin{bmatrix} x_k \\ u_k \end{bmatrix} \tag{D.3a}$$

$$\text{s.t.} \quad x_0 - \hat{x}_0 = 0, \tag{D.3b}$$

$$\lim_{N\to\infty} x_{-N} = 0, \tag{D.3c}$$

$$x_{k+1} - Ax_k - Bu_k = 0, \quad k = \ldots, 0, 1, \ldots \tag{D.3d}$$

$$\lim_{N\to\infty} x_N = 0, \tag{D.3e}$$

which implies $V(x) = V^-(x) + V^+(x)$.

We prove now that V it is a Lyapunov function for the closed-loop system using the MPC feedback from Problem (D.1). First, we assume that the system is controllable, so that $V^-(\hat{x}_0) < \infty$ for all bounded initial values. We will extend the proof to stabilisable systems in a second step.

We begin by proving that $V(x)$ is lower and upper bounded by $\mathcal{K}_\infty$ functions. By assumption, the unique solution of MPC problem (D.1) with initial condition $\hat{x}_0 = 0$ is $x_k = 0$, $u_k = 0$ with a cost $V^+(0) = 0$. Moreover, for $\hat{x}_0 = 0$, Problem (D.3) coincides with Problem (D.1) shifted backwards in time. This entails that, for $\hat{x}_0 = 0$, $x_k^{(D.3)} = 0$, $u_k^{(D.3)} = 0$ must be the unique solution of Problem (D.3a). Therefore, for all $x \neq 0$, any feasible trajectory $\check{x}_k(x)$, $\check{u}_k(x)$ such that $\lim_{N\to\infty} \check{x}_{-N}(x) = 0, \check{x}_0(x) = x$ and $\lim_{N\to\infty} \check{x}_N(x) = 0$ yields a strictly positive cost. Therefore, $V(x) > 0$, for all $x \neq 0$. Controllability implies that $V^+(x) < \infty$ and $V(x) < \infty$. Moreover, because $V(x) \geq 0$ and $V^+(x) < \infty$, it also holds that $V^-(x) > -\infty$. Because the system dynamics are linear, the cost is quadratic and there is no path constraint, $V^+(x)$, $V^-(x)$ and $V(x)$ are quadratic. Then $V(x) > 0$, for all $x \neq 0$ implies that $V(x) = x^\top Wx$, with $W \succ 0$. Therefore,

$$\underline{\alpha}(\|x\|) \leq V(x) \leq \bar{\alpha}(\|x\|),$$

with $\underline{\alpha}$, $\bar{\alpha}$ two $\mathcal{K}_\infty$ functions.

We now turn to prove descent of the Lyapunov function candidate, i.e. for a given initial state $x_0^{(D.1)} = \hat{x}_0 \neq 0$, we must have $V(x_1^{(D.1)}) - V(x_0^{(D.1)}) < 0$. By definition of optimality, we get

$$V^-(x_1^{(D.1)}) \leq V^-(x_0^{(D.1)}) + \frac{1}{2}\begin{bmatrix} x_0^{(D.1)} \\ u_0^{(D.1)} \end{bmatrix}^\top \bar{H} \begin{bmatrix} x_0^{(D.1)} \\ u_0^{(D.1)} \end{bmatrix}, \quad \text{(D.4)}$$

$$V^+(x_0^{(D.1)}) = V^+(x_1^{(D.1)}) + \frac{1}{2}\begin{bmatrix} x_0^{(D.1)} \\ u_0^{(D.1)} \end{bmatrix}^\top \bar{H} \begin{bmatrix} x_0^{(D.1)} \\ u_0^{(D.1)} \end{bmatrix}, \quad \text{(D.5)}$$

and, by replacing (D.5) into (D.4), we obtain

$$V(x_1^{(D.1)}) - V(x_0^{(D.1)}) \leq 0.$$

Because V is bounded and $V(x) > 0$ for all $x \neq 0$, the situation $V(x_1^{(D.1)}) - V(x_0^{(D.1)}) = 0$ can only last at most for a finite number of consecutive steps $n < \infty$, otherwise V would be unbounded. This implies that the feedback from Problem (D.1) is n-step stabilising. However, because the system and stage cost are time invariant and the system is linear, also the feedback from Problem (D.1) is linear time invariant and, therefore, $n = 1$. This means that $V(x_1^{(D.1)}) - V(x_0^{(D.1)}) < 0$ for all $x_0^{(D.1)} \neq 0$, which concludes the first part of the proof.

In case the system is not controllable but stabilisable, the upper bound $V(x) \leq \bar{\alpha}(\|x\|)$ can be violated if x is not reachable from the origin. In order to address that problem, we can formulate a relaxed version of Problems (D.1) and (D.2) which makes use of the relaxed system dynamics $x_{k+1} = Ax_k + Bu_k + \nu_k$ with ν_k a ficticious control penalised by the term $\gamma\|\nu_k\|_1$, with $\gamma \geq \| \lambda_{\max} \|_\infty$, and $\lambda_{\max}$ the Lagrange multiplier of Problems (D.1) and (D.2) whose infinity norm is maximum for all feasible initial conditions. This entails that the relaxed versions of Problems (D.1)-(D.2) yield the same primal solution and optimal value as the original problems for all terminal conditions which are feasible. However, the relaxed problems are feasible for all terminal conditions. Then the proof proceeds along the same arguments used for the controllable case, with the difference that now functions V^+, V^- and V are not quadratic but they are still radially unbounded.□

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