A Partially Tightened Real-Time Iteration Scheme for Nonlinear Model Predictive Control

Andrea Zanelli, Rien Quirynen, Gianluca Frison and Moritz Diehl

Abstract— **In this paper, a strategy is proposed to reduce the computational burden associated with the solution of problems arising in nonlinear model predictive control. The prediction horizon is split into two sections and the constraints associated with the terminal one are tightened using a barrier formulation. In this way, when using the Real-Time Iteration scheme, variables associated with such stages can be efficiently eliminated from the quadratic subproblems by a single backward Riccati sweep. After eliminating the tightened stages, a quadratic problem with a reduced horizon is solved where the original constraints are used. The solution is then expanded to the full horizon with a single forward Riccati sweep. By doing so, the online computational burden associated with the solution of the optimization problems can be largely reduced. Numerical results are reported where, using the proposed scheme, a speedup of about one order of magnitude can be achieved without compromising closed-loop performance.**

I. Introduction

Nonlinear model predictive control (NMPC) is an optimization-based control scheme that has been widely applied in the chemical and process industry since the late 70s. Due to the considerable computational effort required to solve the arising nonlinear, nonconvex optimization problems online, its use in fields where shorter sampling times must be met is still an open challenge. As advances are made in both algorithm development and software implementations, the computation times achievable have shrunk considerably over the past decades. Several works are present in the literature [8], [9] where timings in the milli- and microsecond timescale can be obtained.

In order to reduce the computational burden associated with NMPC, and make it possible for it to be applied to a broader spectrum of applications, among other algorithms, the Real-Time Iteration (RTI) algorithm can be used. When employing such a scheme, a single linearization and a single quadratic program (QP) solve are carried out at each sampling time, leading to an approximate feedback policy. Although the closed-loop performance can be affected, using such an algorithm, a good control performance can often be achieved in practice [1] and closed-loop stability can be guaranteed under the conditions discussed in [4].

Support by the EU via ITN-TEMPO (607957) and ITN-AWESCO (642 682), by DFG via Research Unit FOR 2401, and by BMWi via the project eco4wind is gratefully acknowledged. Andrea Zanelli, Gianluca Frison and Moritz Diehl are with the University of Freiburg, Department of Microsystems Engineering (IMTEK), Georges-Koehler-Allee 102, 79110 Freiburg, Germany - {andrea.zanelli, gianluca.frison, moritz.diehl}@imtek.uni-freiburg.de. Rien Quirynen is with Mitsubishi Electric Research Laboratories, 201 Broadway, Cambridge, MA 02139, USA - quirynen@merl.com.

A. Contributions and Outline

In this paper, an efficient strategy is proposed based on the RTI scheme, which allows one to reduce the computational burden associated with the solution of NMPC problems, by removing part of the constraints in the formulation. In particular, in order to preserve stability and control authority, only the constraints in the late stages of the prediction horizon are removed and replaced by logarithmic barriers. After the linearization step in the RTI, a large part of the variables of the resulting QP can be efficiently eliminated with a backward Riccati sweep. An optimal control problem formulation with a much shorter prediction horizon, and hence fewer variables, is obtained that can be solved with standard QP solvers. Finally, the solution is expanded to the full horizon with a forward sweep. In this way, considerably reduced computation times can be achieved with respect to the standard formulation.

The paper is structured as follows: in Section II, preliminaries on NMPC and the RTI are presented. The algorithm is introduced in Section III and the implementation aspects are discussed in Section IV. In Section V, a stability proof sketch for the proposed method is presented and its potential is assessed on a non-trivial numerical example in Section VI.

II. Preliminaries

A. The Real-Time Iteration Scheme

Consider the following optimal control problem:

$$\begin{aligned} \min_{\substack{x_0,\cdots,x_N \\ u_0,\cdots,u_{N-1}}} \quad & \sum_{i=0}^{N-1} l(x_i,u_i) + l_N(x_N) \\ \text{s.t.} \quad & x_0 - \hat{x}_0 = 0 \\ & x_{i+1} = f(x_i,u_i), \quad i = 0,\cdots,N-1 \\ & g(x_i,u_i) \leq 0, \qquad i = 0,\cdots,N-1 \\ & g_N(x_N) \leq 0, \end{aligned} \tag{1}$$

where

$$\begin{aligned} l(x_i,u_i) &= \frac{1}{2}\left(x_i^T Q x_i + u_i^T R u_i\right) \\ l_N(x_N) &= \frac{1}{2} x_N^T Q_N x_N, \end{aligned} \tag{2}$$

with Q, R and Q_N positive definite and where f, g and g_N are twice continuously differentiable and the inequalities associated with g and g_N define convex sets.

Remark 1: While, from a computational point of view, the proposed algorithm can be applied to a more general problem formulation with potentially nonconvex inequality *constraints, the assumptions above will be made in this work in order to be able to derive preliminary stability results based on the arguments presented in [16].*

A possible approach to solve problem (1) is to use sequential quadratic programming (SQP). SQP-based schemes rely on the solution of a series of quadratic subproblems that locally approximate the original NLP. At every iteration the NLP is linearized and a QP is solved in order to perform an update of the solution. This procedure, combined with additional ingredients required to improve efficiency and guarantee convergence to a local minimum [13], is the main idea used in SQP algorithms.

In order to reduce the computational burden associated with several linearizations and QP solutions at each sampling instant, the RTI scheme can be used. This allows the algorithm to converge over time and achieve faster control feedback to the system. A convergence and stability proof for the RTI scheme is presented in [4] for the simplified setting where no inequality constraints are present.

Although, employing a single SQP step can effectively reduce the computational burden, both linearization and solution of the QP subproblems can give rise to long computation times. An overview on numerical methods for NMPC algorithms can be found in [3].

III. Partially Tightened Optimal Control Problem Formulation

A. Tightening Based on Barrier Functions

A partially tightened formulation will be considered in the following, that allows one to speed up the computations necessary to solve (1). For stages $i = M, \dots, N$, the constraints are removed and the stage costs are replaced with the modified costs

$$\begin{aligned} \tilde{l}(x_i, u_i) &= l(x_i, u_i) + \rho(x_i, u_i), \quad i = M, \dots, N-1 \\ \tilde{l}_N(x_N) &= l_N(x_N) + \rho_N(x_N), \end{aligned} \tag{3}$$

where the logarithmic barriers

$$\begin{aligned} \rho(x_i, u_i) &:= -\tau \sum_{j=1}^{n_g} \log(-g_j(x_i, u_i)) \\ \rho_N(x_N) &:= -\tau \sum_{j=1}^{n_{g_N}} \log(-g_{Nj}(x_N)) \end{aligned} \tag{4}$$

have been introduced. Note that such a formulation is also used in interior-point methods in order to cope with the nonsmoothness of the complementarity conditions. In those methods, the so-called barrier parameter τ is shrunk as the algorithm proceeds in order to obtain a solution to the original problem as $\tau \to 0$ [13].

In the context of model predictive control, strategies that exploit a fixed value of the barrier parameter and tightening of the entire horizon have been investigated by the authors of [16] and [6]. In the first work, for nonlinear systems, the stabilizing properties are described for the feedback policy that is obtained by solving the tightened optimal control problems for a fixed value of the barrier parameter. The second work, for linear systems, presents an approximate strategy that requires, in the limit, performing a single iteration on the subproblems. Convergence and stability of such an algorithm is analyzed in [6]. Finally, the approach in [14] shares some similarities with the work in [16] and [6] in the sense that the C/GMRES iterations can be interpreted as interior-point-like iterations on problems with a particular barrier formulation [3].

B. Partially Tightened Nonlinear OCP Formulation

In the following, a similar approach is used with the important difference that only stages from M to N will be tightened in order not to require a modification of the constraints in the first stages. Moreover, instead of solving exactly the tightened problems, a formulation is proposed that results in QP subproblems with linearized complementarity conditions for the tightened stages. In this way, the variables associated with such stages can be efficiently eliminated with a Riccati-like recursion. After elimination, a smaller QP is obtained, with a reduced horizon that can be efficiently solved with QP solvers tailored to MPC.

It can be easily shown that the first-order optimality conditions associated with problem (1), with stage costs modified according to the barrier formulation in (3) and (4) read:

$$\begin{aligned}
&x_0 - \hat{x}_0 &&= 0 \\
&Qx_0 + \nabla_x f(w_0)\lambda_1 - \lambda_0 - \nabla_x g(w_0)\nu_0 &&= 0 \\
&Ru_0 + \nabla_u f(w_0)\lambda_1 - \nabla_u g(w_0)\nu_0 &&= 0 \\
&g(w_0) + s_0 &&= 0 \\
&S_0\nu_0 &&= 0 \\
&\quad\vdots \\
&x_M - f(w_{M-1}) &&= 0 \\
&Qx_M + \nabla_x f(w_M)\lambda_{M+1} - \lambda_M - \nabla_x g(w_M)\nu_M &&= 0 \\
&Ru_M + \nabla_u f(w_M)\lambda_{M+1} - \nabla_u g(w_M)\nu_M &&= 0 \\
&g(w_M) + s_M &&= 0 \\
&S_M\nu_M &&= \tau\mathbf{1} \\
&\quad\vdots \\
&x_N - f(w_{N-1}) &&= 0 \\
&Q_N x_N - \lambda_N - \nabla_x g(w_N)\nu_N &&= 0 \\
&g(w_N) + s_N &&= 0 \\
&S_N\nu_N &&= \tau\mathbf{1},
\end{aligned} \tag{5}$$

where ν_i, $s_i \geq 0$ are the Lagrange multipliers and slacks, respectively, associated with the stage inequalities and λ_i are the equality constraints multipliers. Moreover, S_i denotes the diagonal matrix having the elements of s_i on its diagonal. The compact notation $w_i := (x_i, u_i)$ has been introduced to indicate the stacked state and control variables for stage i.

C. Real-Time Iterations with Partial Tightening

Let us consider a modified version of the Generalized Gauss-Newton (GGN) based SQP method [2] for the problem in (5) based on a linearization of the nonlinear constraints and of the complementarity conditions for stages M to N:

$$
\begin{aligned}
&x_0 - \hat{x}_0 && = 0 \\
&Qx_0 + A_0^T \lambda_1 - \lambda_0 - G_0^{x^T} \nu_0 && = 0 \\
&Ru_0 + B_0^T \lambda_1 - G_0^{u^T} \nu_0 && = 0 \\
&d_0 + G_0^x x_0 + G_0^u u_0 + s_0 && = 0 \\
&S_0 \nu_0 && = 0 \\
&\quad\vdots && \\
&x_M - A_{M-1} x_{M-1} - B_{M-1} u_{M-1} - c_{M-1} && = 0 \\
&Qx_M + A_M^T \lambda_{M+1} - \lambda_M - G_M^{x^T} \nu_M && = 0 \\
&Ru_M + B_M^T \lambda_{M+1} - G_M^{u^T} \nu_0 && = 0 \\
&d_M + G_M^x x_M + G_M^u u_M + s_M && = 0 \\
&e_M + \bar{S}_M \nu_M + \bar{V}_M s_M && = \tau \mathbf{1} \\
&\quad\vdots && \vdots \\
&x_N - A_{N-1} x_{N-1} - B_{N-1} u_{N-1} - c_{N-1} && = 0 \\
&Q_N x_N - \lambda_N - G_N^{x^T} \nu_N && = 0 \\
&d_N + G_N^x x_N + s_N && = 0 \\
&e_N + \bar{S}_N \nu_N + \bar{V}_N s_N && = \tau \mathbf{1},
\end{aligned}
\tag{6}
$$

where

$$
\begin{aligned}
A_i &:= \nabla_x f(\bar{w}_i)^T, \quad & B_i &:= \nabla_u f(\bar{w}_i)^T \\
G_i^x &:= \nabla_x g(\bar{w}_i)^T, \quad & G_i^u &:= \nabla_u g(\bar{w}_i)^T
\end{aligned}
\tag{7}
$$

have been introduced that represent evaluations of the Jacobians of equality and inequality constraints at the linearization point $\bar{w}_i$. Analogously, the matrices $\bar{S}_i$ and $\bar{V}_i$ are the diagonal matrices having the elements of $\bar{s}_i$ and $\bar{\nu}_i$ on their diagonals. Finally, the vectors

$$
\begin{aligned}
c_i &:= \quad f(\bar{w}_i) - A_i \bar{x}_i - B_i \bar{u}_i \\
d_i &:= \quad g(\bar{w}_i) - G_i^x \bar{x}_i - G_i^u \bar{u}_i \\
e_i &:= \quad - \bar{V}_i \bar{s}_i
\end{aligned}
\tag{8}
$$

have been introduced. Notice that, if an SQP step is applied to the partially tightened problem, where logarithmic barriers are used explicitly in the cost, analogously, a set of linear equations would be obtained for the terminal section of the horizon and the scheme would naturally fall in the standard SQP framework. However, for numerical reasons, it might be convenient instead to use the formulation proposed above. Although different intermediate iterations would be taken by the two schemes in general, if convergence is achieved, a solution to (5) is obtained by both algorithms.

Notice that, when applying the standard RTI scheme, the presence of constraints gives rise to nonsmooth equations that require special treatment. If, for example, the resulting QP is solved with an interior-point method, the complementarity conditions for the first M stages need to be relaxed and iteratively solved within one SQP iteration. In the proposed scheme, the smoothed complementarity conditions for stages $i = M, \dots, N$ have been linearized in (6) such that the computational burden associated with one RTI iteration can be largely reduced. In particular, a single backward Riccati recursion [15] can be used to factorize the part of the KKT matrix associated with the tightened stages. These implementation aspects are discussed further in Section IV.

IV. Efficient Algorithm Implementation

In this section, the implementation of the proposed algorithm is discussed. An efficient Riccati recursion based on the algorithm proposed in [15] is used to eliminate variables associated with the tightened stages in order to obtain, after elimination, a reduced QP with a shorter horizon. Note that the solution of the resulting QP is expanded back into the solution for the original long horizon problem.

A. The Backward Riccati Recursion

To perform an RTI, system (6) needs to be solved, including the positivity constraints for the slack variables and Lagrange multipliers. For stages M to N, it can be shown that the linear system associated with a Newton step has the form of the KKT system that arises from a linear-quadratic problem [15]. In particular, after eliminating slack variables and inequality multipliers, this leads to a system with a special band diagonal structure that can be exploited in order to reduce the computational burden. For example, for $N = 2$ and $M = 0$, these equations would read

$$
\begin{bmatrix}
 & -I & & & & & & \\
-I & \tilde{Q}_0 & & A_0^T & & & & \\
 & & \tilde{R}_0 & B_0^T & & & & \\
 & A_0 & B_0 & & -I & & & \\
 & & & -I & \tilde{Q}_1 & & A_1^T & \\
 & & & & & \tilde{R}_1 & B_1^T & \\
 & & & & A_1 & B_1 & & -I \\
 & & & & & & -I & \tilde{Q}_2
\end{bmatrix}
\begin{bmatrix} \lambda_0 \\ x_0 \\ u_0 \\ \lambda_1 \\ x_1 \\ u_1 \\ \lambda_2 \\ x_2 \end{bmatrix}
= -
\begin{bmatrix} r_{E_0} \\ r_{Sx_0} \\ r_{Su_0} \\ r_{E_1} \\ r_{Sx_1} \\ r_{Su_1} \\ r_{E_2} \\ r_{Sx_2} \end{bmatrix},
\tag{9}
$$

with properly defined matrices $\tilde{Q}_i$ and $\tilde{R}_i$ and residuals $r_{E_i}, r_{S_{x_i}}, r_{S_{u_i}}$ according to the elimination procedure described in [15]. It is possible to factorize the matrix in (9) starting from the block corresponding to stage N using the following Riccati recursion:

$$
P_i = \tilde{Q}_i + A_i^T P_{i+1} A_i + \Sigma_i B_i^T P_{i+1} A_i \tag{10}
$$

and

$$
\begin{aligned}
p_i = \; & r_{Sx_i} + A_i^T (P_{i+1} r_{E_i} + p_{i+1}) \\
& + \Sigma_i (r_{Su_i} + B_i^T P_{i+1} r_{E_i} + B_i^T p_{i+1}),
\end{aligned}
\tag{11}
$$

where

$$
\Sigma_i := -(A_i^T P_{i+1} B_i)(\tilde{R}_i + B_i^T P_{i+1} B_i)^{-1}, \tag{12}
$$

for $i = N-1, \dots, M$ initialized with $P_N = \tilde{Q}_N$.

B. The Reduced QP Subproblem and Forward Expansion

Once the variables associated with stages N to M have been eliminated, the following reduced QP with shorter horizon

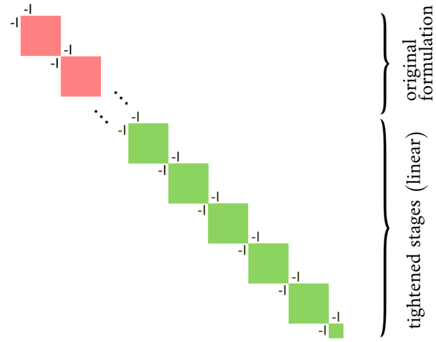

Fig. 1: Backward and forward Riccati recursion. Variables associated with stages M to N can be efficiently eliminated due to the linearized complementarity conditions.

is left to be solved:

$$\begin{aligned}\min_{\substack{x_0,\cdots,x_M\\u_0,\cdots,u_{M-1}}} \quad & \sum_{i=0}^{M-1} l(x_i,u_i) + \psi(x_M) \\ \text{s.t.} \quad & x_0 - \hat{x}_0 = 0 \\ & x_{i+1} - A_i x_i - B_i u_i - c_i = 0, \; i = 0,\cdots,M-1 \\ & d_i + G_i^x x_i + G_i^u u_i \leq 0, \qquad i = 0,\cdots,M-1,\end{aligned} \tag{13}$$

where the terminal cost for stage M is defined as

$$\psi(x_M) := x_M^T P_M x_M + p_M^T x_M, \tag{14}$$

with P_M and p_M both resulting from the backward recursion on stages N to M in (10) and (11).

Remark 2: The procedure described so far can be interpreted as a way of building a terminal cost $\psi(x_M)$ for the MPC problem (13) *with shorter horizon. Compared to the original OCP in* (1)*, $\psi(x_M)$ incorporates an approximate contribution due to stages M to N, which can be efficiently computed thanks to the partially tightened formulation that is adopted.*

The resulting QP in (13) can be readily solved with a standard QP solver tailored to MPC such as qpOASES [7], HPMPC [11] or FORCES PRO [5]. Finally, after solving the QP subproblem, the following recursion is used to update the solution for the tightened stages

$$\begin{aligned} u_i \quad &= K_i x_i + k_i, \\ x_{i+1} \quad &= A_i x_i + B_i u_i + r_{E_i} \\ \lambda_{i+1} \quad &= P_{i+1} x_i + \lambda_i, \end{aligned} \tag{15}$$

for $i = M,\cdots,N-1$, where

$$\begin{aligned} K_i \quad &= -\Gamma_i \, B_i^T P_{i+1} A_i, \\ k_i \quad &= -\Gamma_i \, (r_{Su_i} + B_i^T (P_{i+1} r_{E_i} + p_{i+1})) \\ \Gamma_i \quad &= (\tilde{R}_i + B_i^T P_{i+1} B_i)^{-1}. \end{aligned} \tag{16}$$

Once the solution $z^\star := (x^\star, u^\star, \lambda^\star, \nu^\star, s^\star)$ to equations (6) has been computed, an inexact backtracking line-search is used to obtain a feasible step:

$$z^{k+1} = z^k + t\Delta z, \tag{17}$$

where $\Delta z := z^\star - z^k$ and $t \in (0,1]$ such that the positivity constraints on ν_i and s_i are satisfied. Notice that the positivity constraints for stages 0 to $M-1$ are always satisfied, if a feasible solution to the reduced QP is obtained. Hence, for a practical implementation, the line-search might be limited to the tightened stages. The proposed scheme is summarized in Algorithm 1 and the special block-banded structure with linear stages is described in Figure 1.

Remark 3: Notice that $\hat{x}_0$ is only necessary at line 10 of Algorithm 1. For this reason, when splitting the computations in preparation and feedback phase according to [4], the feedback phase consists only of the solution of the reduced QP, and forward expansion which can further reduce the feedback delays.

Algorithm 1 Partially Tightened Real-Time Iteration

1: **input:** $z^k := (x^k, u^k, \nu^k, \lambda^k, s^k)$, τ
2: **linearization of** (5)**:**
3: compute A_i, B_i, c_i $i = 0,\ldots,N-1,$
4: $\quad G_i, d_i, e_i$ $i = 0,\ldots,N,$
5: $\quad \bar{S}_i, \bar{V}_i,$ $\quad i = M,\ldots,N,$
6: **reduction to equality constrained form** $(N \to M)$:
7: eliminate $s_i^\star, \nu_i^\star$, $i = M,\ldots,N$ according to [15]
8: **backward Riccati sweep** $(N \to M)$:
9: compute P_M and p_M using (10) and (11)
10: **estimate new initial state** $\hat{x}_0$
11: **QP solution** (13)**:**
12: compute $x_i^\star, \lambda_i^\star$ $\quad i = 0,\ldots,M$
13: and $u_i^\star, \nu_i^\star$ $\quad i = 0,\ldots,M-1$
14: **forward Riccati sweep** $(M \to N)$:
15: compute $x_i^\star, \lambda_i^\star$, $i = M,\ldots,N$
16: and $u_i^\star$, $i = M,\ldots,N-1$ using (15) and (16)
17: **final expansion** $(M \to N)$:
18: compute $s_i^\star, \nu_i^\star$, $i = M,\ldots,N$ according to [15]
19: **line-search:**
20: compute z^{k+1} using (17)
21: **output:** z^{k+1}

V. Towards Closed-loop Stability Guarantees

In this section, a stability proof sketch for the proposed scheme is derived which builds on existing results from barrier function-based NMPC [16]. The derivations in the following extend the results to the case in which the original constraints are kept in the formulation for stages 0 to $M-1$ and, successively, to the case where a single Newton-type step is applied to the resulting optimality conditions, based on the arguments used in [4]. In order to be able to use the results from [16], the following assumptions will be made:

Assumption 1: For the logarithmic barriers associated with the stage constraints defined in (4) the following holds:

$$\begin{aligned}&\rho(0,0) = 0\\&\rho(x,u) > 0, \; \forall (x,u) \in \Omega := \{(x,u) : g(x,u) < 0\},\end{aligned}$$

and, analogously, for the terminal constraint

$$\begin{aligned}&\rho_N(0) = 0\\&\rho_N(x) > 0, \; \forall x \in \Omega_N := \{x : g_N(x) < 0\}.\end{aligned}$$

Assumption 2: The terminal constraint function $g_N(x_N)$ is quadratic

$$g_N(x_N) = x_N^T H_N x_N - \alpha_N,$$

for some positive definite matrix H_N and value $\alpha_N > 0$.

Notice that, if Assumption 1 does not hold, the so-called *gradient recentred* barrier functions can be used as proposed in [16]. Given the Assumption 1 and 2, the following result holds for the case where constraints are tightened throughout the entire horizon, i.e. if $M = 0$:

Lemma 4 ([16]): Consider the optimal control problem in (1) with all the stage costs modified according to (3) and (4). For any value of $\tau > 0$, there exist suitable Q_N, H_N and α_N such that the following decrease property holds:

$$-\tilde{l}_N(x_N) + \tilde{l}_N(f(x_N, Kx_N)) + \tilde{l}(x_N, Kx_N) \leq 0, \quad (18)$$

for all $x_N \in \Omega_N$, where $u = Kx$ defines an asymptotically stabilizing control law around the origin.

A detailed discussion, based on arguments from [12], can be found in [16]. In particular, a way to select H_N and Q_N based on properties of gradient recentred self-concordant barrier functions is proposed. Lemma 4 provides the descent properties required to construct a Lyapunov function for a fully tightened problem. Consider now a partially tightened problem, in which the original quadratic cost is used for stages $i = 0, \ldots, M$. The candidate Lyapunov function

$$V^*(\hat{x}_0) := \sum_{i=0}^{M-1} l(x_i^*, u_i^*) + \sum_{i=M}^{N-1} \tilde{l}(x_i^*, u_i^*) + \tilde{l}_N(x_N^*) \quad (19)$$

is taken into account.

Theorem 5: For a proper choice of H_N and Q_N according to the results in [16] and $\forall M \in \{0, \ldots, N\}$, the following inequality holds for any $\hat{x}_0$ for which (5) admits solutions:

$$-V^*(\hat{x}_0) + V^*(f(\hat{x}_0, u_0^*)) \leq -l(\hat{x}_0, u_0^*). \quad (20)$$

Proof: The proof uses standard arguments based on the construction of a feasible, but suboptimal, trajectory obtained by shifting the optimal solution computed for $\hat{x}_0$. Let $\hat{V}(f(\hat{x}_0, u_0^*))$ denote the cost associated with such a suboptimal trajectory and define

$$\Delta\hat{V} := -V^*(\hat{x}_0) + \hat{V}(f(\hat{x}_0, u_0^*)). \quad (21)$$

The following equality holds by construction of the suboptimal policy obtained by shifting:

$$\begin{aligned}\Delta\hat{V} \quad &= -l(\hat{x}_0, u_0^*) - \tilde{l}_N(x_N) + \tilde{l}_N(f(x_N, Kx_N)) \\ &\quad + \tilde{l}(x_N, Kx_N) + l(x_M, u_M) - \tilde{l}(x_M, u_M).\end{aligned} \quad (22)$$

Note that the following holds

$$l(x_M, u_M) - \tilde{l}(x_M, u_M) = \tau \sum_{i=1}^{n_g} \log(-g_j(x_M, u_M)) < 0,$$

where Assumption 1 has been used. Due to the decrease property in Lemma 4, the following holds:

$$\Delta\hat{V} \leq -l(\hat{x}_0, u_0^*). \quad (23)$$

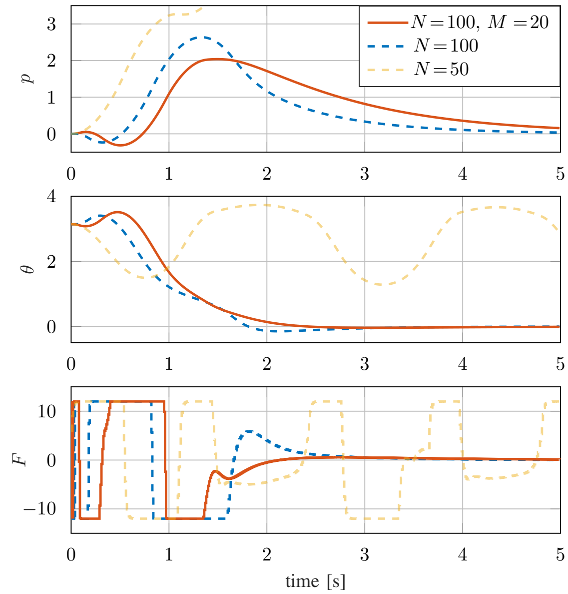

Fig. 2: Inverted pendulum swing-up: comparison of RTI schemes. Partially tightened RTI scheme (bold solid red) with $N = 100$, $M = 20$ and $\tau = 1$, standard formulation with $N = 100$ (dashed blue) and with $N = 50$ (dashed yellow). The partially tightened formulation stabilizes the system keeping the original constraints only in the initial 20 stages. The same does not hold for a standard formulation with a shortened horizon of $N = 50$ nodes.

Moreover, as the shifted policy used to derive such an inequality is suboptimal, after optimizing, the following must hold:

$$-V^*(\hat{x}_0) + V^*(f(\hat{x}_0, u_0^*)) \leq -l(\hat{x}_0, u_0^*). \quad (24)$$

■

Although a rigorous stability analysis is out of the scope of this work and still under development, the result derived shows that when controlling a system using a solution to the set of nonlinear equations (5), under mild assumptions, certain descent properties for $V^*(\hat{x}_0)$ can be provided. However, the proposed scheme solves, at every iteration, the linearized system (6). In order to extend the stability proof sketch, arguments from [4] can be used. In particular, the proof assumes that a neighborhood around the origin exists, where the active-set is stable. Notice that it can be shown that such an assumption is satisfied if the second order sufficient conditions for optimality hold [13]. Then, the nonlinear root-finding problem (5) falls in the class covered by the stability proof for the standard RTI scheme without shifts. Hence, the results from [4] can be applied to the proposed algorithm where no inequality constraints are present, e.g when $M = 0$. A more detailed analysis of the stability properties of this new variant of the RTI scheme, especially in the presence of inequality constraints and active set changes, is part of ongoing research.

VI. Numerical Case Study

In the following, the presented approach will be validated on a nontrivial example. Consider the following ordinary differential equations describing the dynamics of an inverted

	$N=100$	$M=50$	$M=20$	$M=10$	$M=5$	$M=2$
QP [ms]	4.405	2.571	1.034	0.566	0.313	**0.237**
lin. [ms]	0.190	0.190	0.190	0.190	0.190	0.190
total [ms]	4.595	2.761	1.224	0.756	0.503	**0.427**
i.p. iter.	37	32	29	26	17	17
c.l. cost	1037	1005	1160	1128	1338	1288
$\text{speedup}_{\text{QP}}$	–	1.71	4.26	7.78	14.07	**18.59**
$\text{speedup}_{\text{RTI}}$	–	1.66	3.75	6.07	9.13	**10.76**

TABLE I: Pendulum example: worst-case computation times for the swing-up closed-loop scenario (1000 sampling steps) in milliseconds and closed-loop cost with $N = 100$, $\tau = 1$ and decreasing values of M (standard RTI with $N = 100$ in the first column).

pendulum:

$$\begin{aligned} \dot{p} = v, \quad \dot{v} &= \frac{-lms_\theta\omega^2 + F + gmc_\theta s_\theta}{M + m - mc_\theta^2} \\ \dot{\theta} = \omega, \quad \dot{\omega} &= \frac{-lmc_\theta s_\theta\omega^2 + Fc_\theta + gms_\theta + Mgs_\theta}{l(M + m - mc_\theta^2)} \end{aligned} \tag{25}$$

where $x = (p, \theta, v, \omega)$ is the state of the system, where p and v are the linear position and velocity of the cart and θ and ω are the angle and angular velocity of the pendulum. The input to the system is the force F applied to the cart, while m, M, l and g are fixed parameters representing the mass of the pendulum, the mass of the cart, the length of the pendulum and gravitational acceleration respectively. The notation $s_\theta := \sin(\theta)$ and $c_\theta := \cos(\theta)$ is used.

An OCP of the form in (1) is considered, where $f(\cdot)$ represents the discretized dynamics obtained by applying the explicit Runge-Kutta scheme of order four with fixed step-size $h = 0.01s$. A control horizon $T = 1s$ is used and the trajectories are discretized using $N = 100$ shooting nodes. Simple bounds are imposed on the input $F_{\max} = -F_{\min} = 12\,N$ and the cost matrices have been chosen as follows:

$$Q = \text{diag}(1 \cdot 10^{-1},\, 1,\, 1 \cdot 10^{-1},\, 2 \cdot 10^{-3}) \quad \text{and} \quad R = 5 \cdot 10^{-4},$$

and an LQR-based terminal cost is used.

The proposed scheme has been implemented using the open-source interior-point solver HPMPC [11] that exploits high-performance linear algebra package BLASFEO [10]. Figure 2 shows the closed-loop trajectories obtained with three different setups. An RTI scheme with standard formulation is used with $N = 50$ and $N = 100$ and the proposed partially tightened formulation is used with $M = 20$, $N = 100$ and $\tau = 1$. The partially tightened scheme stabilizes the system, keeping the original form of the constraints only in few stages. In this way, a long prediction horizon can be used while reducing the computational burden associated with the iterations with respect to the standard RTI scheme. If a shorter horizon of e.g. $N = 50$ shooting nodes is used, the system cannot be stabilized in this case. Timing results are reported in Table I for the inverted pendulum swing-up example, using a varying number of untightened stages M and $\tau = 1$. For small values of M, a large speedup can be achieved with respect to the standard formulation with $N = 100$.

VII. Conclusions

An efficient partially tightened RTI scheme for NMPC has been presented. The algorithm uses a barrier formulation to approximate stage constraints in the terminal part of the prediction horizon. In this way a large part of the variables in the QP subproblems can be eliminated with a single backward Riccati recursion sweep. After solving a reduced QP for the initial part of the horizon, the solution is expanded back to the full horizon. A stability proof sketch is derived and a numerical case study is presented that shows both closed-loop simulations and detailed timing results.

References

[1] T. Albin, D. Ritter, N. Liberda, R. Quirynen, and M. Diehl. In-vehicle realization of nonlinear MPC for gasoline two-stage turbocharging airpath control. *IEEE Transactions on Control Systems Technology*, pages 1–13, 2017.

[2] H. G. Bock. Recent advances in parameter identification techniques for ODE. In *Numerical Treatment of Inverse Problems in Differential and Integral Equations*, pages 95–121. Birkhäuser, 1983.

[3] M. Diehl, H. J. Ferreau, and N. Haverbeke. Efficient numerical methods for nonlinear MPC and moving horizon estimation. In L. Magni, M.D. Raimondo, and F. Allgöwer, editors, *Nonlinear model predictive control*, volume 384 of *Lecture Notes in Control and Information Sciences*, pages 391–417. Springer, 2009.

[4] M. Diehl, R. Findeisen, and F. Allgöwer. A stabilizing real-time implementation of nonlinear model predictive control. In L. Biegler, O. Ghattas, M. Heinkenschloss, D. Keyes, and B. van Bloemen Waanders, editors, *Real-Time and Online PDE-Constrained Optimization*, pages 23–52. SIAM, 2007.

[5] A. Domahidi, A. Zgraggen, M. N. Zeilinger, M. Morari, and C. N. Jones. Efficient interior point methods for multistage problems arising in receding horizon control. In *Proceedings of the IEEE Conference on Decision and Control (CDC)*, pages 668–674, Maui, HI, USA, December 2012.

[6] Christian Feller and Christian Ebenbauer. A stabilizing iteration scheme for model predictive control based on relaxed barrier functions. *arXiv preprint at arXiv:1603.04605*, April 2016.

[7] H. J. Ferreau, C. Kirches, A. Potschka, H. G. Bock, and M. Diehl. qpOASES: a parametric active-set algorithm for quadratic programming. *Mathematical Programming Computation*, 6(4):327–363, 2014.

[8] Hans Joachim Ferreau, Boris Houska, Kurt Geebelen, and Moritz Diehl. Real-time control of a kite-model using an auto-generated nonlinear MPC algorithm. In *Proceedings of the 18th IFAC World Congress*, 2011.

[9] J. V. Frasch, A. J. Gray, M. Zanon, H. J. Ferreau, S. Sager, F. Borrelli, and M. Diehl. An auto-generated nonlinear MPC algorithm for real-time obstacle avoidance of ground vehicles. In *Proceedings of the European Control Conference (ECC)*, pages 4136–4141, 2013.

[10] G. Frison, D. Kouzoupis, T. Sartor, A. Zanelli, and M. Diehl. BLASFEO: Basic linear algebra subroutines for embedded optimization. *arXiv:1704.02457*, 2017.

[11] G. Frison, H. B. Sorensen, B. Dammann, and J. B. Jørgensen. High-performance small-scale solvers for linear model predictive control. In *Proceedings of the European Control Conference (ECC)*, pages 128–133, June 2014.

[12] D. Q. Mayne, J. B. Rawlings, C. V. Rao, and P. O. M. Scokaert. Constrained model predictive control: Stability and optimality. *Automatica*, 26(6):789–814, 2000.

[13] J. Nocedal and S. J. Wright. *Numerical Optimization*. Springer Series in Operations Research and Financial Engineering. Springer, 2 edition, 2006.

[14] T. Ohtsuka. A tutorial on C/GMRES and automatic code generation for nonlinear model predictive control. In *Control Conference (ECC), 2015 European*, pages 73–86, July 2015.

[15] C. V. Rao, S. J. Wright, and J. B. Rawlings. Application of interior-point methods to model predictive control. *Journal of Optimization Theory and Applications*, 99:723–757, 1998.

[16] A. G. Wills and W. P. Heath. Barrier function based model predictive control. *Automatica*, 40(8):1415–1422, August 2004.