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A Homotopy-based Nonlinear Interior-Point Method for NMPC

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Abstract: This paper introduces a homotopy-based nonlinear interior-point method that can exploit warm-starts for an efficient real-time implementation of nonlinear model predictive control (NMPC). The algorithm performs a homotopy on a tightened problem with a fixed value of the barrier parameter during which the initial state is changed gradually. Once an approximate solution to the tightened problem is obtained, a second homotopy is performed that shrinks the barrier parameter in order to compute a solution to the original problem. Theoretical results are presented on the local convergence, which provide a second order contraction estimate for both phases of the algorithm. In order to assess the potential of the proposed scheme, it has been implemented in the software package FORCES NLP. Its performance on a non-trivial NMPC case study is shown, where a speedup of up to one order of magnitude is obtained.

Keywords: nonlinear model predictive control, numerical methods, embedded optimization, nonlinear interior-point methods.

1. INTRODUCTION

Nonlinear model predictive control (NMPC) has drawn increasing attention in both academia and industry in the past decades. Due to its inherent capability to handle multivariable constrained nonlinear systems directly, it is a promising control strategy in several fields (Qin and Badgwell, 2000). In addition, theoretical results that provide stability guarantees for NMPC schemes under reasonable assumptions have been developed (Mayne et al., 2000). However, due to the high computational burden associated with the online solution of the nonlinear and in general nonconvex optimal control problems (OCP), NMPC has been historically employed mainly in the chemical industry, where the sampling times are generally sufficiently long (García et al., 1989; Qin and Badgwell, 2003).

More recently, as more efficient algorithms are being developed and more powerful embedded computing units are becoming available, computation times in the milli- and microsecond time-scale have been achieved for optimal control problems arising in applications in the fields of automotive (Frasch et al., 2013), renewable energy (Ferreau et al., 2011) and robotics (Diehl et al., 2006).

When using NMPC to control a system, a sequence of closely related, parametric optimization problems has to be solved online, each for a different initial state value. This fact can be exploited to warm-start the algorithm when solving successive instances of the nonlinear optimal control problem. Indeed, efficient online methods such as the Real-Time Iteration (RTI) scheme (Diehl et al., 2007) and the Continuation GMRES (C/GMRES) algorithm (Ohtsuka, 2004) use warm-starting and continuation to track the optimal solution manifold.

Although sequential quadratic programming (SQP) methods are well known to be able to exploit warm-starts, this is less straightforward for interior-point methods (Gondzio and Grothey, 2006). Several issues associated with warm-starts are discussed and potential solutions are proposed in the literature. In (Yildirim and Wright, 2002), for linear programs, two strategies are proposed based on adjustments of iterates available from previously solved neighboring problems. It is shown that, for a sufficiently large value of the barrier parameter, it is possible to obtain a feasible point for the perturbed problem applying such corrections to the stored iterates. Similarly, in (Gondzio and Grothey, 2006), unblocking heuristics are presented that can improve performance when solving linear programs. In (Shahzad et al., 2010) and (Shahzad and Goulart, 2011), a warm- and a hot-start strategy for interior-point methods are presented, respectively, with applications to linear model predictive control. For nonlinear nonconvex problems, in (Benson and Shanno, 2008), a warm-starting technique is proposed based on a penalty approach.

1.1 Contributions and Outline

This paper proposes an interior-point method for nonlinear nonconvex problems arising from NMPC formulations that can exploit warm-starts. Similarly to what is proposed in (Gondzio and Grothey, 2006) and (Yildirim and Wright, 2002), the algorithm uses an intermediate iterate on a previously solved neighboring problem. The algorithm consists of two separate phases. During the initial phase, a homotopy is performed on a tightened problem with a

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fixed value of the barrier parameter, which can exploit a warm-start from previous problem instances. Once an approximate solution to the initial barrier problem is obtained, a second homotopy is performed in which the barrier parameter is decreased in order to compute an optimal solution to the original optimization problem. By storing the primal-dual solution to the initial barrier problem, a warm-start for the next OCP instance is made available. The local convergence of this continuation technique is analyzed and a contraction estimate is derived for the proposed algorithm. The presented scheme is implemented in the software package FORCES NLP (Zanelli et al., 2016) that uses a structure-exploiting interior-point method for multi-stage nonlinear nonconvex optimization. The potential of this approach for NMPC applications is assessed on a non-trivial example, where a speedup of up to one order of magnitude can be achieved.

The paper is organized as follows: Section 2 presents the preliminaries on NMPC and the problem formulation. In Section 3, the algorithm is described and the theoretical results are derived and discussed in Section 4. Finally, Section 5 presents the numerical case study.

2. PRELIMINARIES

2.1 Nonlinear Model Predictive Control

Throughout the paper, the following discrete-time optimal control problem will be considered:

$$\begin{aligned} \min_{\substack{x_0,\dots,x_N \\ u_0,\dots,u_{N-1}}} \quad & \sum_{i=0}^{N-1} l_i(x_i,u_i) + l_N(x_N) \\ \text{s.t.} \quad & x_0 - \hat{x}_0 = 0 \\ & x_{i+1} = f(x_i,u_i), \quad i = 0,\dots,N-1 \\ & g(x_i,u_i) \le 0, \quad i = 0,\dots,N-1 \\ & g_N(x_N) \le 0, \end{aligned} \tag{1}$$

where the functions l_i, l_N, f, g and g_N are twice continuously differentiable. States and inputs of the dynamical system are represented by x and u, respectively. When using NMPC to control a system, problem (1) has to be solved, at every sampling instant, for a new $\hat{x}_0$ describing the current state of the system. The first optimal input u_0^* is applied to the system and, at the next sampling instant, a new optimization problem is solved. In this way, the control algorithm can compensate for model uncertainty and disturbances.

2.2 Newton-type Optimization

Problem (1) can be solved in several ways. In particular, two main classes of methods can be identified (Nocedal and Wright, 2006): the so-called sequential quadratic programming (SQP) methods and interior-point methods. The first relies on the solution of a series of convex quadratic programs (QP) that locally approximate the original problem. The second directly solves the relaxed Karush-Kuhn-Tucker (KKT) system associated with problem (1):

$$\begin{aligned} \nabla_x l_0(w_0) + \nabla_x f(w_0)\lambda_1 - \lambda_0 - \nabla_x g(w_0)\nu_0 &= 0 \\ \nabla_u l_0(w_0) + \nabla_u f(w_0)\lambda_1 - \nabla_u g(w_0)\nu_0 &= 0 \\ x_0 - \hat{x}_0 &= 0 \\ g(w_0) + s_0 &= 0 \\ S_0\nu_0 &= \tau\mathbf{1} \\ \vdots \qquad\qquad & \vdots \\ \nabla_x l_N(x_N) - \lambda_N &= 0 \\ x_N - f(w_{N-1}) &= 0 \\ g(x_N) + s_N &= 0 \\ S_N\nu_N &= \tau\mathbf{1}, \end{aligned} \tag{2}$$

where S_i is the diagonal matrix having the elements of the slack variable s_i on its diagonal and $\mathbf{1}$ denotes a vector of ones. The compact notation $w_i := [\, x_i^T \; u_i^T \,]^T$ has been introduced. The barrier parameter τ has been introduced in order to circumvent the nonsmoothness of the complementarity conditions. The equations in (2) for $\tau = 0$, together with the positivity conditions $s \ge 0$ and $\nu \ge 0$, constitute the so-called first-order necessary optimality conditions.

The Newton method can then be directly applied to (2) for decreasing values of τ and, as $\tau \to 0$, a point that satisfies first-order optimality conditions is recovered. Moreover, under mild assumptions (Nocedal and Wright, 2006), the iterates converge to a local minimum. Practical implementations of interior-point methods generally include additional algorithmic ingredients, to ensure global convergence and improve performance and reliability (Wächter and Biegler, 2006), (Vanderbei, 1999).

2.3 Predictor-corrector Methods

In the following, problem (2) is referred to in compact form

$$F(z) + C\xi = 0, \tag{3}$$

where z is the vector of stacked primal, dual and slack variables and $\xi := (\hat{x}_0, \tau)$. An approximate solution to (3) for a given $(\hat{x}_0, \tau)$ will be denoted by $\tilde{z}(\hat{x}_0, \tau) \approx \bar{z}(\hat{x}_0, \tau)$, where $\bar{z}(\hat{x}_0, \tau)$ denotes the exact solution. Note that an exact Newton step (Deuflhard, 2011) reads as

$$z^{k+1} = z^k - \frac{\partial F}{\partial z}(z^k)^{-1}\left(F(z^k) + C\xi\right), \tag{4}$$

for a given z^k and ξ. As discussed in (Tran-Dinh et al., 2012), a predictor-corrector step can be performed to obtain an approximate solution $z^{k+1} \approx \bar{z}(\xi^{k+1})$ for a new parameter value ξ^{k+1}, given $z^k \approx \bar{z}(\xi^k)$. This combined predictor-corrector step takes the form

$$\begin{aligned} z^{k+1} &= z^k - \frac{\partial F}{\partial z}(z^k)^{-1}\left(F(z^k) + C\xi^k\right) \\ &\quad - \frac{\partial F}{\partial z}(z^k)^{-1} C\left(\xi^{k+1} - \xi^k\right) \\ &= z^k - \frac{\partial F}{\partial z}(z^k)^{-1}\left(F(z^k) + C\xi^{k+1}\right). \end{aligned} \tag{5}$$

As the parameter ξ enters linearly in Eq. (3), the above predictor-corrector step corresponds to a standard Newton step (4) applied directly to the problem with the new value ξ^{k+1}. This concept of introducing the parameter ξ linearly in order to simplify the continuation procedure is also referred to as parameter embedding in (Diehl, 2002).

Note that, instead of using the exact Jacobian $J(z^k) := \frac{\partial F}{\partial z}(z^k)$ and its factorization in (5), it is common practice to use a Jacobian approximation $M_k \approx J(z^k)$ in order to reduce the overall computational burden (Deuflhard, 2011). We further consider the general class of Newton-type methods, e.g., including Gauss-Newton or quasi-Newton Hessian approximations.

3. THE HOMOTOPY-BASED ALGORITHM

When solving neighboring instances of (1) for different initial conditions $\hat{x}_0$ with an interior-point method, information available from previously computed solutions is difficult to exploit for warm-starting in general. If an approximate solution $\tilde{z}(\hat{x}_0, 0)$ to equation (3) is used to initialize the Newton-type iterates for a different $\hat{x}_0$, a number of problems can arise (Benson and Shanno, 2008). These issues are generally related to the need to bring the barrier parameter to some positive value τ_0 in order to be able to adjust the iterates, without violating the positivity constraints imposed on inequality multipliers ν and slacks s. In particular, certain components of ν and s in $\tilde{z}(\hat{x}_0, 0)$ might be very close to zero and the enforcement of the positivity constraints can lead to very small steps. Moreover, due to the presence of the complementarity conditions, equations (3) become highly nonlinear for small values of τ, resulting in a poor performance of Newton-type algorithms.

Algorithm 1 Homotopy method: Phase I

input: $\xi^0 = (\hat{x}_0^-, \tau_0)$, $z^0 = \tilde{z}(\xi^0)$, $\hat{x}_0^+$
for $k = 0, \dots, k_1^{\max}$
  $\hat{x}_0^{k+1} \leftarrow \phi_1(\xi^k, z^k)$
  $\xi^{k+1} \leftarrow (\hat{x}_0^{k+1}, \tau_0)$
  $z^{k+1} \leftarrow z^k - M_k^{-1}\left(F(z^k) + C\xi^{k+1}\right)$
  if $\left\|F(z^{k+1}) + C\xi^{k+1}\right\| < \epsilon_1$ and $\hat{x}_0^{k+1} = \hat{x}_0^+$
    return z^{k+1}
  end
end

In order to be able to exploit solutions available from previous instances of the OCP, a homotopy-based interior-point method is proposed in the following. Assume that problem (3) for $\hat{x}_0 = \hat{x}_0^-$ has been previously solved and a new solution has to be computed for $\hat{x}_0 = \hat{x}_0^+$. The main additional ingredient with respect to a standard warm-starting procedure, where $\tilde{z}(\hat{x}_0^-, 0)$ would be used, is the use of an approximate solution $\tilde{z}(\hat{x}_0^-, \tau_0)$ for some $\tau_0 > 0$. The idea consists in splitting the iterates on the relaxed problem (3) into two different phases. In the first phase, Newton-type iterates are performed on (3) for $\tau = \tau_0$ and $\hat{x}_0$ approaching $\hat{x}_0^+$, starting from $\hat{x}_0^-$. This configuration might be interpreted as the Newton-type method being applied to a smooth, unconstrained problem and initialized with $z^0 = \tilde{z}(\hat{x}_0^-, \tau_0)$. For modest perturbations of $\hat{x}_0$, the iterates would quickly converge to a solution of the intermediate barrier problems. Algorithm 1 summarizes this first phase of the scheme, which is additionally illustrated in Figure 1.

Once an approximate solution to (3) for $\tau = \tau_0$ and $\hat{x}_0 = \hat{x}_0^+$ has been obtained, the algorithm switches to a second phase in which τ is shrunk according to a standard barrier strategy for interior-point methods. Before doing so, the intermediate solution $\tilde{z}(\hat{x}_0^+, \tau_0)$ is stored in order to be able to warm-start the algorithm for the next instance. This final procedure is described in Algorithm 2 and illustrated in Figure 1. Note that the performance of both Algorithm 1 and 2 depends on the policy used to update the homotopy parameter ξ, which is respectively defined by the functions $\phi_1(\cdot)$ and $\phi_2(\cdot)$. Advanced strategies can be used for this purpose, which are known in the context of path-following or continuation methods. A detailed discussion on the design of these update policies is outside the scope of this paper and the interested reader is referred to (Allgower and Georg, 1990; Deuflhard, 2011).

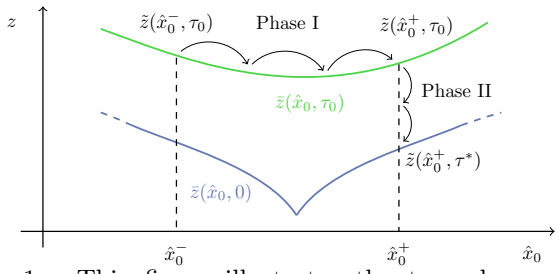

Fig. 1. This figure illustrates the two phases of the homotopy-based interior-point algorithm. During Phase I, the barrier parameter $\tau = \tau_0$ is kept fixed and the initial state of the system $\hat{x}_0$ is updated. Once the approximate solution $\tilde{z}(\hat{x}_0^+, \tau_0)$ is obtained, Phase 2 is used to compute a solution to the original problem.

Algorithm 2 Homotopy method: Phase II

input: $\xi^0 = (\hat{x}_0^+, \tau_0)$, $z^0 = \tilde{z}(\xi^0)$
for $k = 0, \dots, k_2^{\max}$
  $\tau^{k+1} \leftarrow \phi_2(\xi^k, z^k)$
  $\xi^{k+1} \leftarrow (\hat{x}_0^+, \tau^{k+1})$
  $z^{k+1} \leftarrow z^k - M_k^{-1}\left(F(z^k) + C\xi^{k+1}\right)$
  if $\left\|F(z^{k+1}) + C\xi^{k+1}\right\| < \epsilon_1$ and $\tau^{k+1} < \tau^*$
    return z^{k+1}
  end
end

Remark 1: In Algorithm 1, the initial state $\hat{x}_0$ is potentially updated several times according to a given policy $\phi_1(\cdot)$. However, in NMPC applications with sufficiently high sampling rates, it might be sufficient to perform the homotopy from $\hat{x}_0^-$ to $\hat{x}_0^+$ in one iteration of Phase I. The numerical results in Section 5 will indeed use this approach to solve a series of neighboring optimization problems in an NMPC implementation.

Remark 2: Notice that Algorithms 1 and 2 can be seen as special cases of a single algorithm in which the homotopy parameter ξ is updated, according to some policy ϕ. In particular, as both $\hat{x}_0$ and τ enter the problem linearly, the analysis presented in the following section holds for both algorithms. On the one hand, it provides convergence results for Phase I during which $\hat{x}_0$ is updated. On the other hand, the same results apply to Phase II, where the barrier parameter τ is changed.

4. LOCAL CONVERGENCE RESULTS

In this section, theoretical results regarding the convergence of the proposed method are derived. In particular, we consider the Newton-type predictor-corrector step in line 5 of Algorithm 1 and 2. As $\hat{x}_0$ and τ enter the problem linearly, the following results hold for both Phase I and Phase II of the algorithm.

4.1 Local Contraction Theorem

For a compact notation, the solution to (3) for $\xi = \xi^k$ will be referred to as $\bar{z}^k$. Consider the Newton-type update

$$z^{k+1} = z^k - M_k^{-1}(F(z^k) + C\xi^{k+1}), \tag{6}$$

where a Jacobian approximation M_k is used in (5). Notice that, at the solution $\bar{z}^{k+1}$, the following holds:

$$\bar{z}^{k+1} = \bar{z}^{k+1} - M_k^{-1}(F(\bar{z}^{k+1}) + C\xi^{k+1}), \tag{7}$$

because $F(\bar{z}^{k+1}) + C\xi^{k+1} = 0$.

Assumption 1. (Lipschitz continuity). There exists a constant $\sigma \geq 0$ such that for every solution $\bar{z}^k$ and $\bar{z}^{k+1}$, associated with ξ^k and ξ^{k+1}, respectively, the following inequality holds:

$$\left\|\bar{z}^{k+1} - \bar{z}^k\right\| \leq \sigma \left\|\xi^{k+1} - \xi^k\right\|.$$

Assumption 2. (ω- and κ-conditions). There exist $\omega < \infty$ and $\kappa < 1$ such that, for any given solution $\bar{z}^k$, iterate z^k and iteration matrix M_k satisfy

(1) $\left\|M_k^{-1}(J(z^k) - M_k)\right\| \leq \kappa$

(2) $\left\|M_k^{-1}(J(z) - J(\bar{z}^k))\right\| \leq \omega \left\|z - \bar{z}^k\right\|, \forall z.$

The following Theorem provides a convergence proof for the Newton-type predictor-corrector method in (6), based on the results in (Tran-Dinh et al., 2012).

Theorem 3. Let Assumptions 1 and 2 hold. The following inequality holds for the sequence $\left(z^k\right)_{k\geq 0}$, generated by the Newton-type predictor-corrector iterations (6):

$$\begin{aligned}\|\Delta z_{k+1}\| \leq (\kappa + \omega\sigma \left\|\xi^{k+1} - \xi^k\right\| + \frac{\omega}{2}\|\Delta z_k\|)\|\Delta z_k\| \\ + (\kappa\sigma + \frac{\omega\sigma^2}{2}\left\|\xi^{k+1} - \xi^k\right\|)\left\|\xi^{k+1} - \xi^k\right\|,\end{aligned} \tag{8}$$

where $\Delta z_{k+1} := z^{k+1} - \bar{z}^{k+1}$ and $\Delta z_k := z^k - \bar{z}^k$.

Proof. Using (6) and (7), Δz_{k+1} can be rewritten as

$$\begin{aligned}\Delta z_{k+1} &= M_k^{-1}(M_k z^k - F(z^k) - C\xi^{k+1}) \\ &\quad - M_k^{-1}(M_k \bar{z}^{k+1} - F(\bar{z}^{k+1}) - C\xi^{k+1}),\end{aligned}$$

adding and subtracting $M_k^{-1}\left(M_k\bar{z}^k + F(\bar{z}^k)\right)$

$$\begin{aligned}\Delta z_{k+1} &= M_k^{-1}(M_k(z^k - \bar{z}^k) - F(z^k) + F(\bar{z}^k)) \\ &\quad - M_k^{-1}(M_k(\bar{z}^{k+1} - \bar{z}^k) - F(\bar{z}^{k+1}) + F(\bar{z}^k)) \\ &= M_k^{-1}M_k(z^k - \bar{z}^k) - M_k^{-1}M_k(\bar{z}^{k+1} - \bar{z}^k) \\ &\quad - M_k^{-1}\int_0^1 J(\bar{z}^k + t(z^k - \bar{z}^k))(z^k - \bar{z}^k)\mathrm{d}t \\ &\quad + M_k^{-1}\int_0^1 J(\bar{z}^k + t(\bar{z}^{k+1} - \bar{z}^k))(\bar{z}^{k+1} - \bar{z}^k)\mathrm{d}t,\end{aligned}$$

adding and subtracting $M_k^{-1}J(z^k)(z^k + \bar{z}^{k+1} - 2\bar{z}^k)$

$$\begin{aligned}\Delta z_{k+1} &= M_k^{-1}(M_k - J(z^k))(z^k - \bar{z}^k) \\ &\quad - M_k^{-1}(M_k - J(z^k))(\bar{z}^{k+1} - \bar{z}^k) \\ &\quad - M_k^{-1}\int_0^1 (J(\bar{z}^k + t(z^k - \bar{z}^k)) - J(z^k))(z^k - \bar{z}^k)\mathrm{d}t \\ &\quad + M_k^{-1}\int_0^1 (J(\bar{z}^k + t(\bar{z}^{k+1} - \bar{z}^k)) - J(z^k))(\bar{z}^{k+1} - \bar{z}^k)\mathrm{d}t.\end{aligned}$$

Then, using the κ- and ω-conditions in Assumption 2

$$\begin{aligned}\|\Delta z_{k+1}\| &\leq \kappa\left\|z^k - \bar{z}^k\right\| + \kappa\left\|\bar{z}^{k+1} - \bar{z}^k\right\| \\ &\quad + \omega\int_0^1 \left\|\bar{z}^k + t(z^k - \bar{z}^k) - z^k\right\|\mathrm{d}t\left\|z^k - \bar{z}^k\right\| \\ &\quad + \omega\int_0^1 \left\|\bar{z}^k + t(\bar{z}^{k+1} - \bar{z}^k) - z^k\right\|\mathrm{d}t\left\|\bar{z}^{k+1} - \bar{z}^k\right\| \\ &\leq (\kappa + \frac{\omega}{2}\|\Delta z_k\|)\|\Delta z_k\| \\ &\quad + (\kappa + \frac{\omega}{2}\left\|\bar{z}^{k+1} - \bar{z}^k\right\| + \omega\|\Delta z_k\|)\left\|\bar{z}^{k+1} - \bar{z}^k\right\|\end{aligned}$$

and, finally, due to the regularity Assumption 1

$$\begin{aligned}\|\Delta z_{k+1}\| &\leq (\kappa + \omega\sigma\left\|\xi^{k+1} - \xi^k\right\| + \frac{\omega}{2}\|\Delta z_k\|)\|\Delta z_k\| \\ &\quad + (\kappa\sigma + \frac{\omega\sigma^2}{2}\left\|\xi^{k+1} - \xi^k\right\|)\left\|\xi^{k+1} - \xi^k\right\|.\end{aligned}$$

■

Theorem 3 shows how the update of the parameter ξ affects contraction of the Newton-type iterates. In particular, it can be expected that, for small enough perturbations, the optimal manifold can be tracked by the predictor-corrector scheme.

4.2 An Illustrative Example

In order to illustrate the results of Theorem 3, consider the nonlinear root-finding problem (3) with

$$F(z) := \begin{bmatrix} 20z_1z_2 - z_2 \\ \log(3z_2 + 3) + \sin(z_1) + \log(z_1) \end{bmatrix} \tag{9}$$

and $C = 3 \cdot I_2$. Figure 2 shows the results obtained by applying the homotopy method, with exact Jacobians, to Eq. (9) for 50 iterations with

$$\xi_k = \xi_0 + \frac{(\xi_f - \xi_0)\cdot k}{50}, \quad k = 1, \ldots, 50,$$

where $\xi_0 = [0\ 0]^T$ and $\xi_f = [5\ 5]^T$. Every 5 iterations, the parameter is frozen and five exact Newton steps are taken to illustrate the locally quadratic convergence.

5. NUMERICAL RESULTS

In the following, the computational advantages of the proposed method are assessed on a non-trivial numerical example. The nonlinear nonconvex optimization problems arising from an NMPC formulation will be solved with both a cold-started and a homotopy-based interior-point method. For this purpose, the presented homotopy-based algorithm has been implemented in the software package FORCES NLP (Zanelli et al., 2016) that uses a structure-exploiting primal-dual interior-point method for multi-stage nonlinear nonconvex problems.

The benchmark consists in the swing-up of an inverted pendulum, described by the differential equations:

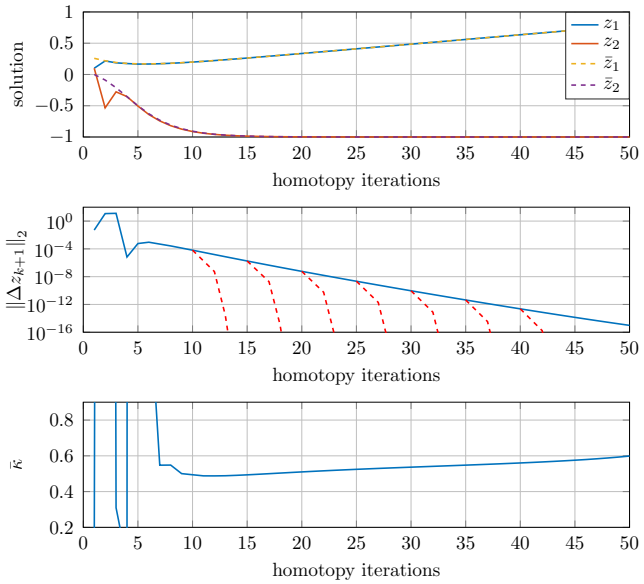

Fig. 2. Convergence of the homotopy method on the simple root-finding problem described by (9). Every five Newton iterations, ξ is fixed to illustrate the locally quadratic convergence (dashed red). The quantity $\bar{\kappa} := \frac{\|\Delta z_{k+1}\|_2}{\|\Delta z_k\|_2}$ shows the contraction rate of the iterations, where $\Delta z_k := z^k - \bar{z}^k$.

$$\dot{x} = \begin{bmatrix} v \\ \omega \\ \dfrac{-lms_\theta\omega^2 + F + gmc_\theta s_\theta}{M + m - mc_\theta^2} \\ \dfrac{-lmc_\theta s_\theta\omega^2 + Fc_\theta + gms_\theta + Mgs_\theta}{l(M + m - mc_\theta^2)} \end{bmatrix}, \qquad (10)$$

where $x = (p, \theta, v, \omega)$ is the state of the system, in which p and v are the linear position and velocity of the cart and θ and ω are the angle and angular velocity of the pendulum. The input to the system is the force F applied to the cart, while m, l, M and g are fixed parameters representing the mass of the pendulum, its length, the mass of the cart, and gravity respectively. Note that the compact notation $s_\theta := \sin(\theta)$ and $c_\theta := \cos(\theta)$ is used in Eq. (10).

An OCP of the form in (1) is considered, where $f(\cdot)$ represents the discretized dynamics obtained by applying the explicit Runge-Kutta scheme of order four with fixed step-size and ten intermediate integration steps. A control horizon $T = 2s$ is used and the trajectories are discretized using $N = 70$ shooting nodes (Bock and Plitt, 1984). Simple bounds are imposed on the input:

$$-10\,N \leq F \leq 10\,N \qquad (11)$$

and a quadratic cost

$$l_i(x_i, u_i) = \frac{1}{2}\left(x_i^T Q x_i + u_i^T R u_i\right) + q^T x_i + r^T u_i$$
$$l_N(x_N) = \frac{1}{2} x_N^T Q_N x_N + q_N^T x_N,$$

has been used, with

$$Q = \mathrm{diag}(\,0.5,\ 1,\ 0.002,\ 0.002), \qquad R = 0.001$$

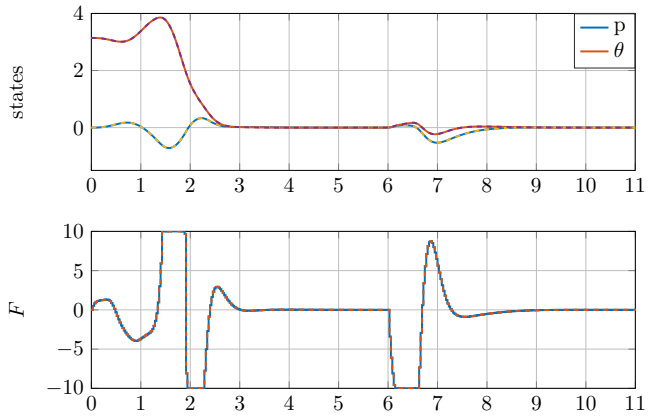

Fig. 3. Closed-loop trajectories for the pendulum example: both the cold-started and the homotopy-based algorithm result in the same swing-up performance.

and $q = -Qx_r$, $r = 0$ and $q_N = -Q_N x_r$. The reference $x_r(t)$ is defined as follows:

$$x_r(t) = \begin{cases} [0\ \pi\ 0\ 0]^T & t \leq 2s \\ [0\ 0\ 0\ 0]^T & t > 2s. \end{cases}$$

The proposed homotopy-based implementation uses a fixed $\tau_0 = 0.0001$, a single step in Phase I and a monotone strategy (Fiacco and McCormick, 1990) in Phase II. The cold-started algorithm uses instead an adaptive barrier strategy that, at every step, adjusts τ according to a measure of progress on the complementarity condition (Vanderbei, 1999). Notice that, for the cold-started algorithm, the adaptive strategy has been chosen over the monotone one, since the latter led to poor performance in the numerical experiments. Both implementations use a blocked BFGS Hessian approximation described in (Bock and Plitt, 1984).

Figure 3 shows the closed-loop trajectories obtained with FORCES NLP using both algorithms, while Figure 4 compares the two strategies in terms of number of iterations required to solve the optimization problems and timings. Note that a disturbance is applied to the system between time $t_1 = 6.0s$ and $t_2 = 6.2s$ by replacing the optimal input u_0^* with the perturbed input $u_0^* + 8$. Using the homotopy-based approach the number of iterations can be largely reduced, leading to considerable speedups. The worst-case number of iterations of 280 is reduced to 57 iterations and, for a considerable part of the scenario, a speedup of more than an order of magnitude is achieved.

6. CONCLUSIONS AND OUTLOOK

A homotopy-based interior-point algorithm for NMPC applications that can exploit warm-starts has been proposed. Theoretical results that provide a contraction estimate for the algorithm are derived. In particular, it is shown that the optimal manifold can be tracked with a single Newton-type iteration for every homotopy parameter update. The method has been implemented in the software package FORCES NLP and the advantages of the algorithm have been assessed on a non-trivial example where a speedup of more than an order of magnitude can be achieved with respect to a cold-started implementation.

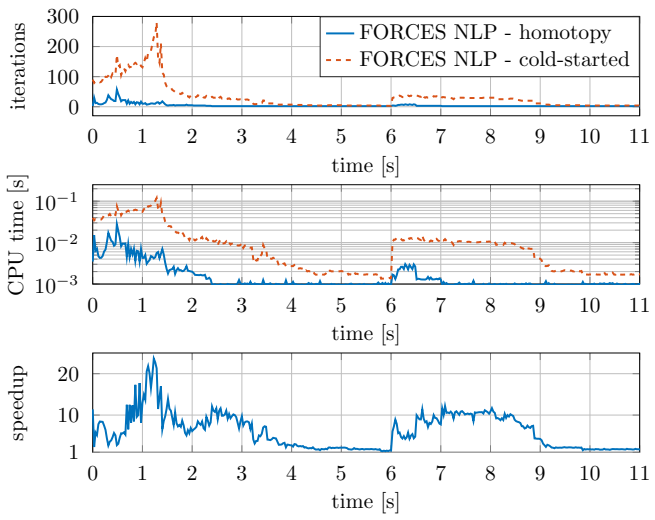

Fig. 4. Number of iterations and computation times for the cold-started and homotopy-based interior-point method, based on a BFGS Hessian approximation. The number of iterations can be largely reduced, resulting in a speedup of about an order of magnitude.

An additional implementation of the proposed approach that exploits the hardware-tailored linear algebra routines available in the software package HPMPC (Frison et al., 2014) is part of ongoing development.

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